Chapter 9: Impulse and Momentum

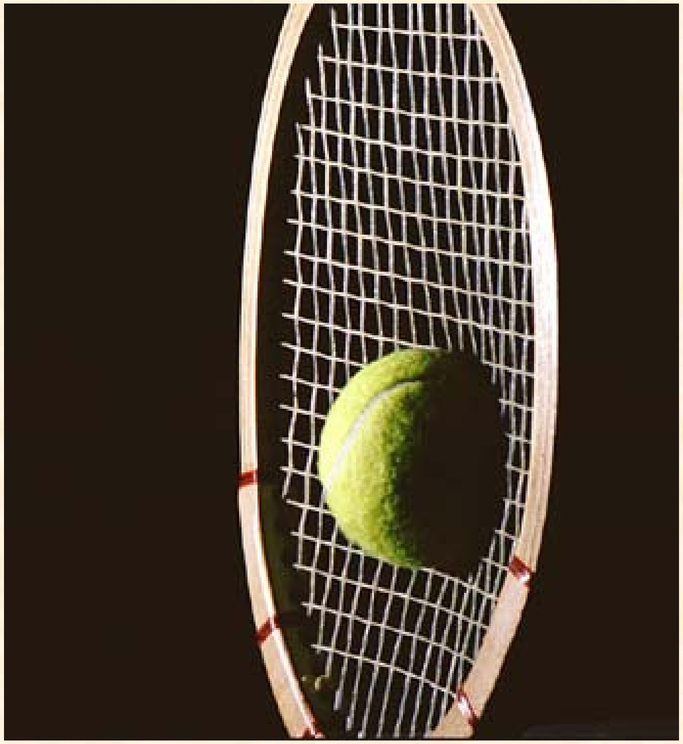
Midterm: covers everything in chapters 1-8

- three problems, each worth 10 points.
  - **first problem is actually five short-answer (1 line) questions (definitions, F = this, a = that, what is m?)**
  - **second and third problems are medium-length worked out problems, about a page or less each.**
  - **worth 15% of the course grade**
- Surnames A-E: Loyola 191, surnames F-Z: Sobey 160. Please be in your assigned room by 9:55 am, Tuesday, October 28.
- bring pencils and an eraser. ***No notes or electronic devices of any kind:*** paper, calculators, and formulae provided.

Evidence of a visit from the Great Pumpkin himself will be found by all!

Chapter 9: Impulse and Momentum

9.1 Impulse-momentum theorem



A collision between a ball and a racquet is an example in physics where relatively simple "before" and after" states (*e.g.*, how fast the ball is going) are separated by an enormously complicated event (*e.g.*, the collision).

Is there a way to understand how the before and after states are related to each other without having to worry about the messy details of the collision?

You bet there is!

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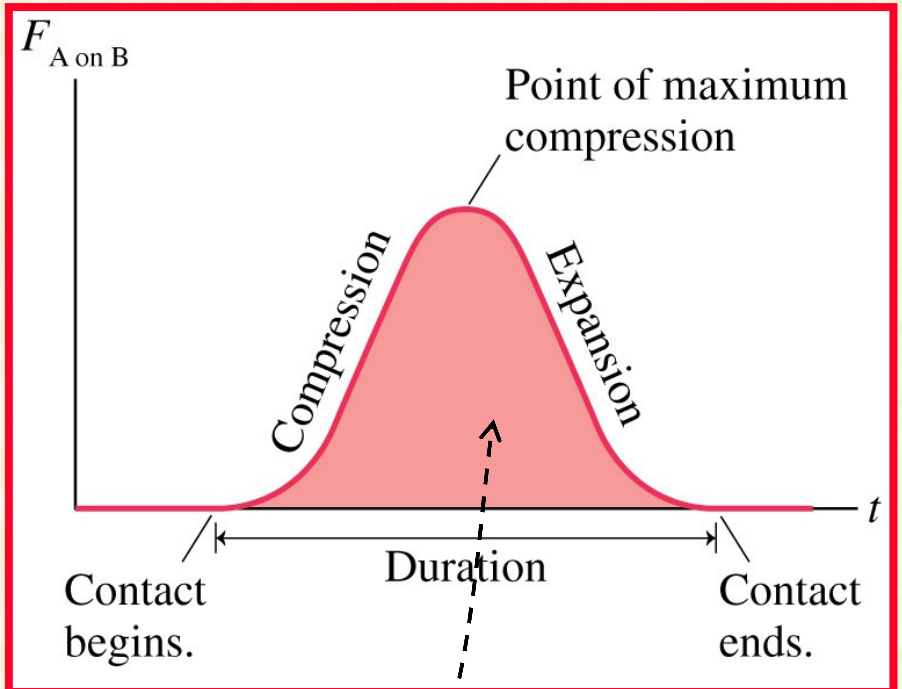
Even the simplest model of a tennis ball compressing then expanding against a racquet shows that during the collision, the force is ***variable*** with a short duration.



If the force on the ball is in the x-direction, we have:

$$F_x(t) = m a_x = m\frac{dv_x}{dt} \quad \Rightarrow \quad \underbrace{m\int_{v_i}^{v_f} dv_x}_{mv_{fx} - mv_{ix} \;=\; p_{fx} - p_{ix}} = \underbrace{\int_{t_i}^{t_f} F_x(t)\,dt}_{\text{area under the } F_x(t)\ vs.\ t \text{ graph} \;\equiv\; J_x}$$

$mv_{fx} - mv_{ix} = p_{fx} - p_{ix}$ (p_x is the **momentum**)

area under the $F_x(t)$ *vs.* t graph $\equiv J_x$ (**impulse**)

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$$\vec{p} = m\vec{v} \qquad \text{(momentum)}$$

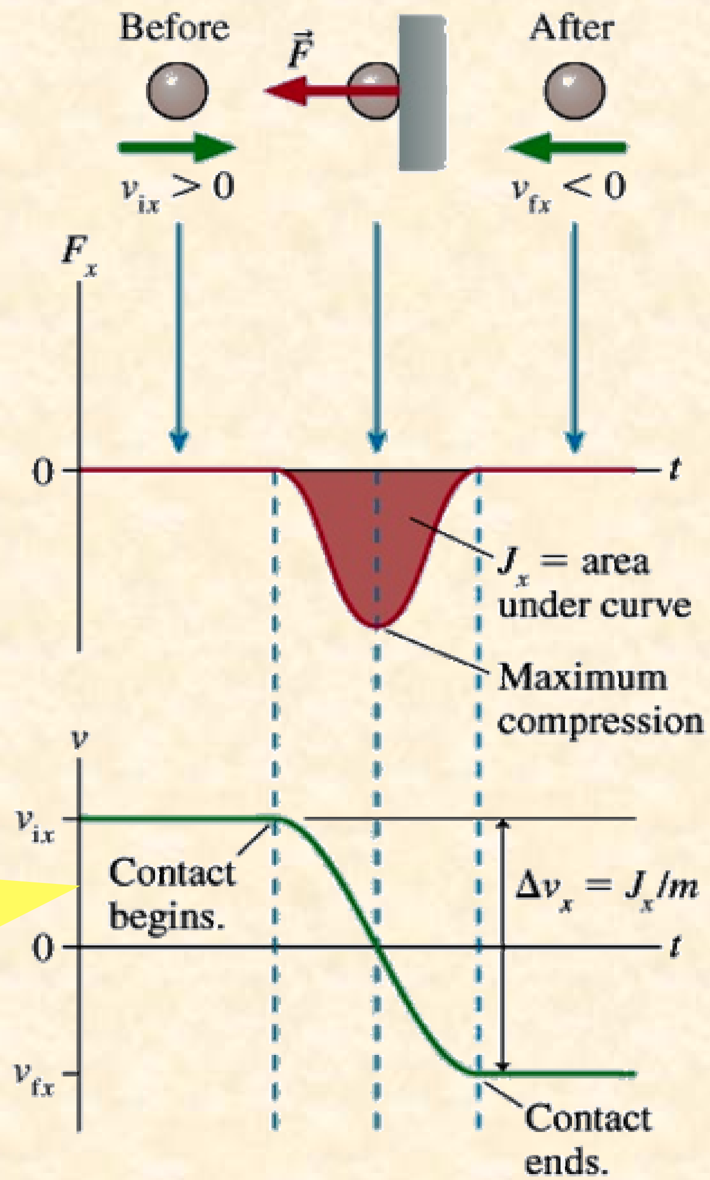
$$\vec{J} = \int_{t_i}^{t_f} \vec{F}dt \qquad \text{(impulse)}$$

$$\vec{p}_f - \vec{p}_i = \boxed{\Delta\vec{p} = \vec{J}} = \int_{t_i}^{t_f} \vec{F}(t)dt$$

Boxed in yellow is the "impulse-momentum theorem", and is a restatement of Newton's 2nd Law.

units of $\vec{p}$: kg m s^{-1}
units of $\vec{J}$: N s (= kg m s^{-1})

An impulse changes an object's momentum (*e.g.*, the tennis ball). **Note the change in sign in v_x, and thus p_x.**



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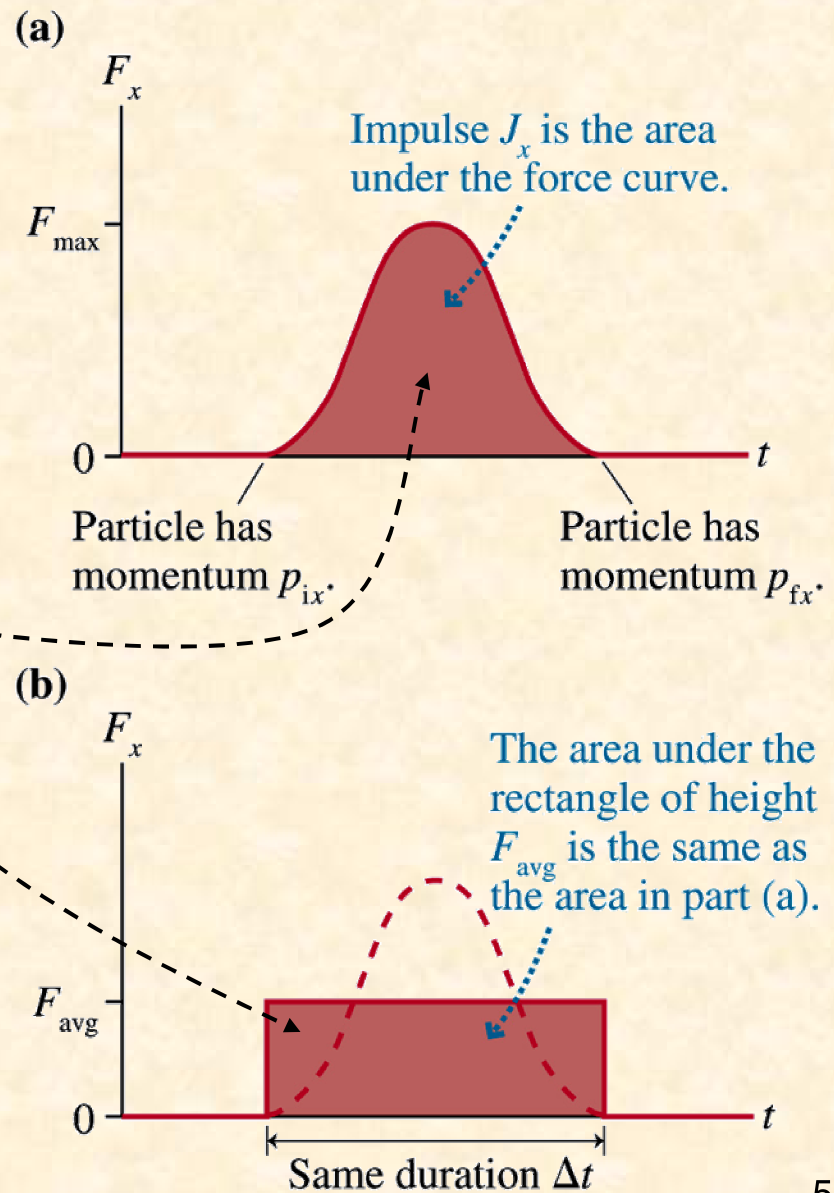
The ***average force***, F_{avg}, over the duration of the interaction, $\Delta t = t_f - t_i$, is defined to be such that:

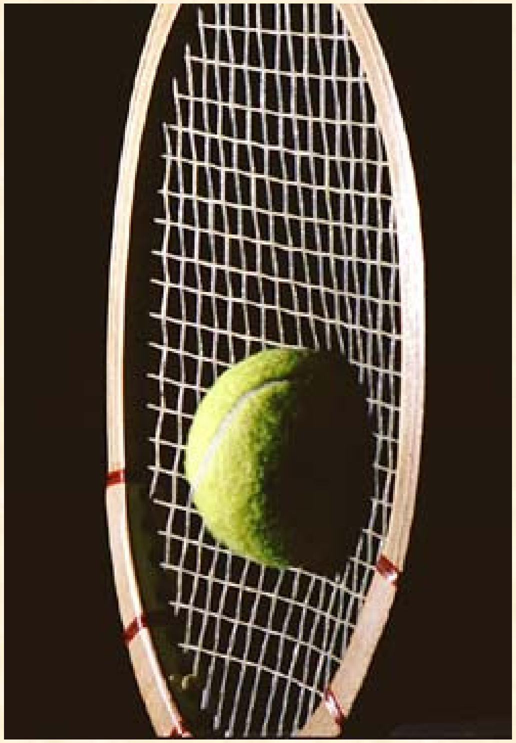
$$\underbrace{F_{avg}\,\Delta t}_{\text{area of rectangle}} \equiv \underbrace{\int_{t_i}^{t_f} F_x(t)\,dt}_{\text{area under curve}} = J_x$$

$$\Rightarrow \boxed{F_{avg} = \frac{J_x}{\Delta t}}$$

This is actually an application of the "mean value theorem" in calculus (for those who know what that is ☺).



Chapter 9: Impulse and Momentum



Let's put some numbers to this.

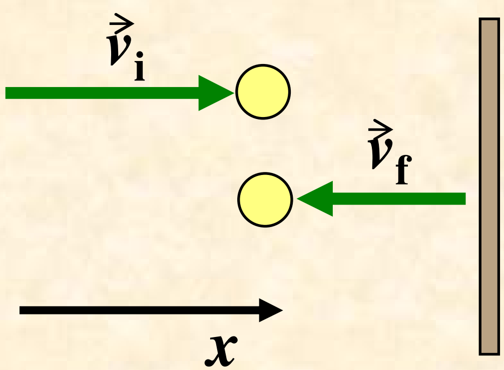
Among the hardest hitters in tennis was "Big Bill" Tilden who, in 1931, was recorded hitting the ball at 232 kph! (73 ms^{-1})

Assuming your racquet survives the impact, what impulse does your racquet have to provide to return Bill's volley with 2/3 of the speed intact? (The mass of a standard tennis ball is 58 g.)

$$p_{ix} = mv_{ix} = (0.058)(73.0) = 4.23\ \mathrm{kg\,m\,s^{-1}}$$

$$p_{fx} = mv_{fx} = -(0.058)\frac{2}{3}(73.0) = -2.82\ \mathrm{kg\,m\,s^{-1}}$$

$$\Rightarrow\ \Delta p_x = p_{fx} - p_{ix} = -7.06\ \mathrm{kg\,m\,s^{-1}}$$



Thus, the racquet must exert an impulse of 7.06 Ns. If the duration of the collision is 0.002 s, the average force is 7.06/0.002 = 3540 N ~ 800 lbs!

Clicker question 9.1

The cart's change in momentum is:

a) –30 $\mathrm{kg\,m\,s^{-1}}$

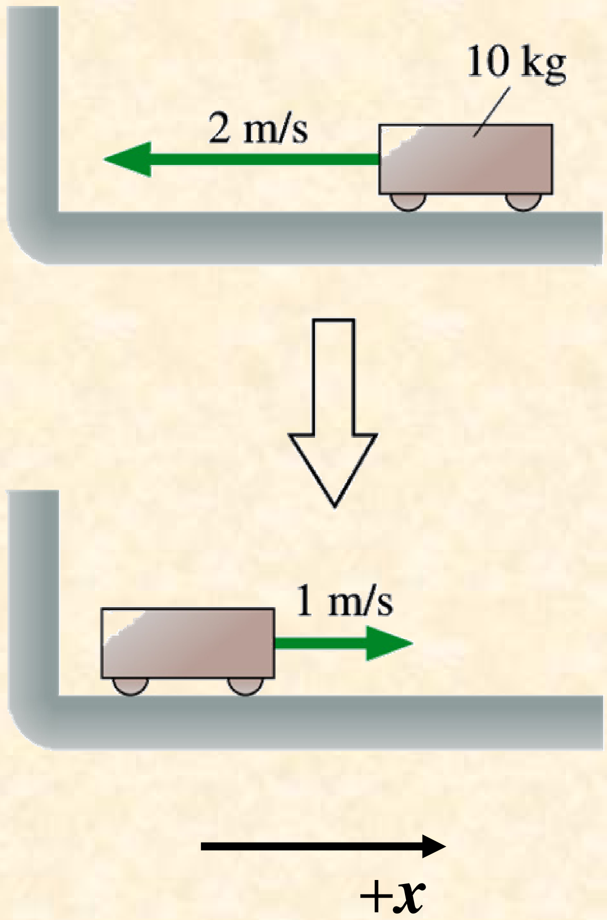
b) –20 $\mathrm{kg\,m\,s^{-1}}$

c) –10 $\mathrm{kg\,m\,s^{-1}}$

d) 10 $\mathrm{kg\,m\,s^{-1}}$

e) 20 $\mathrm{kg\,m\,s^{-1}}$

f) 30 $\mathrm{kg\,m\,s^{-1}}$



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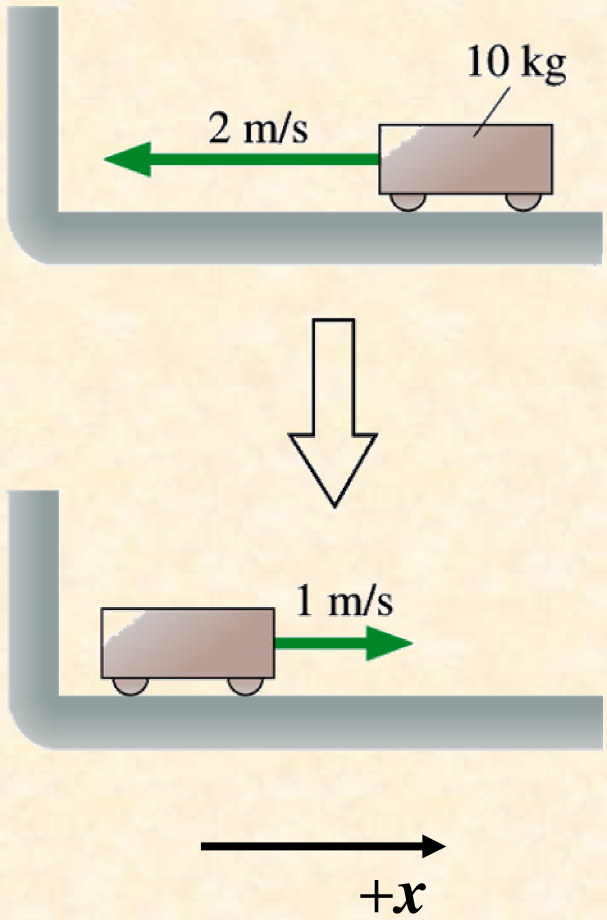
e) 20 kg m s^{-1}

f) 30 kg m s^{-1} ✔

$$\Delta p_x = m(v_{fx} - v_{ix}) = 10[1 - (-2)] = 30 \text{ kg m s}^{-1}$$



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Clicker question 9.2

A rubber ball and a clay ball, each with the same mass, are thrown at a wall with equal speeds. The rubber ball bounces off the wall, while the clay ball sticks to it. Which ball exerts a larger impulse on the wall?

a) The clay ball exerts a larger impulse because it sticks.

b) The rubber ball exerts a larger impulse because it bounces.

c) They exert equal impulses against the wall because they started off with the same momenta.

d) Neither ball exerts any impulse against the wall because the wall doesn't move.

Chapter 9: Impulse and Momentum

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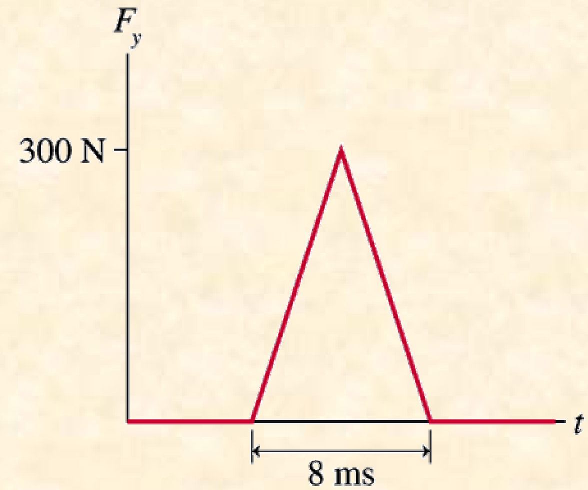
Example. A 100 g rubber ball is dropped from a height of 2.0 m onto a hard floor, with the force exerted by the floor on the ball shown in the figure. How high does the ball bounce?



On the way down…

$$v_{1y}^2 = v_{0y}^2 - 2g\Delta y \quad \Rightarrow \quad v_{1y} = \sqrt{-2(9.8)(-2.0)} = \underline{-6.26\ \text{ms}^{-1}} \text{ (ball moving down)}$$

Impulse from floor:

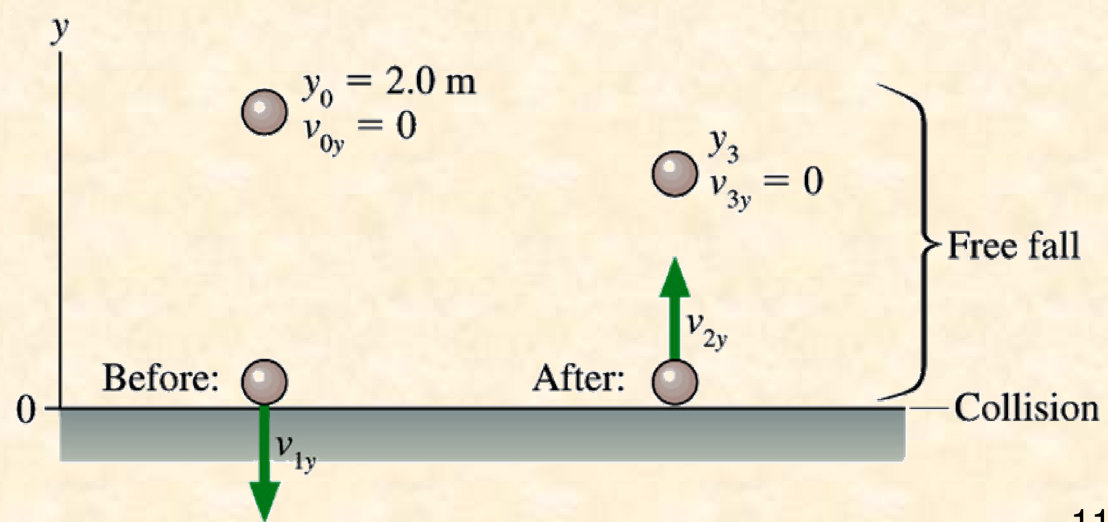
J = area under $F(t)$ *vs.* t plot

$= \frac{1}{2}(0.008)(300) = 1.2$ Ns

$= \Delta p_y = p_{2y} - p_{1y} = m(v_{2y} - v_{1y})$

$\Rightarrow \quad 1.2 = 0.1[v_{2y} - (-6.26)]$

$\Rightarrow \quad v_{2y} = 5.74\ \text{ms}^{-1}$



Chapter 9: Impulse and Momentum

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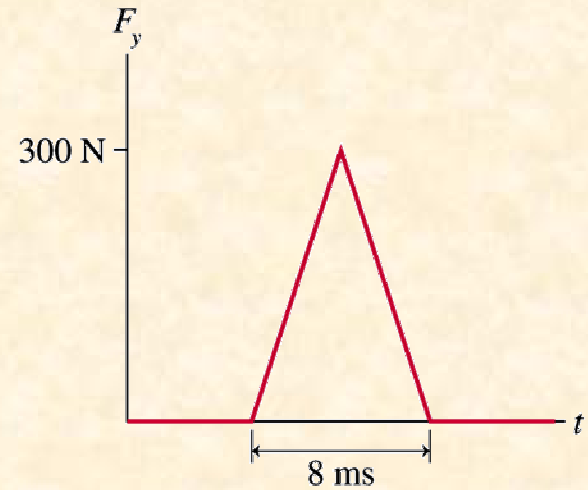

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$$\Rightarrow \quad 1.2 = 0.1[v_{2y} - (-6.26)]$$

$$\Rightarrow \quad v_{2y} = 5.74\ \text{ms}^{-1}$$

On the way up…

$$v_{3y}^2 = v_{2y}^2 - 2g\Delta y = 0$$

$$\Rightarrow \quad \Delta y = \frac{v_{2y}^2}{2g} = \underline{1.68\ \text{m}}$$

Chapter 9: Impulse and Momentum

In the previous example:

We neglected the ball's weight during the collision with the floor. Why? Doesn't it affect the momentum of the ball too?

Yes it does, BUT, over the short duration of the collision, the impulse of the gravitational force is

$\boldsymbol{J_{mg}} = \boldsymbol{mg\Delta t} = 0.100(9.8)(0.008) = 0.0078 \text{ Ns} \ll \boldsymbol{J}_{\text{floor}} = 1.2 \text{ Ns}.$

Neglecting small impulses caused by other forces is called **the impulse approximation**.

The impulse approximation is the most accurate when used to evaluate velocities **immediately** before and after collision. Then use kinematics/dynamics for the rest of the problem.

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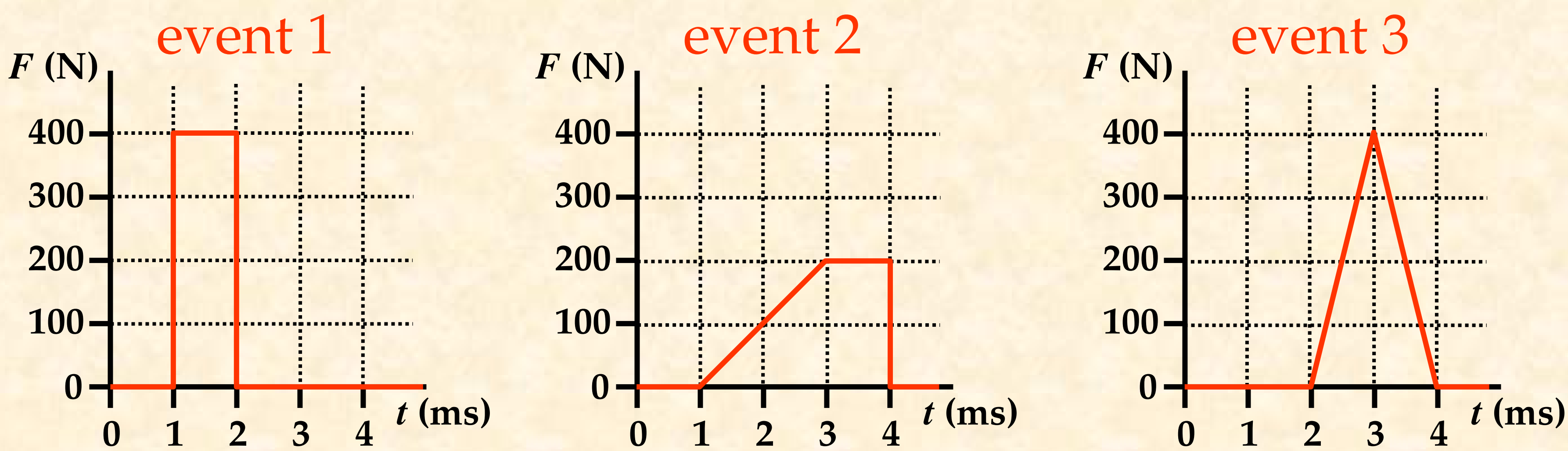
Clicker question 9.3 In three separate interactions, object A exerts a time-dependent force over a relatively short time on object B, as represented in the ***F***(***t***) vs. ***t*** graphs below.

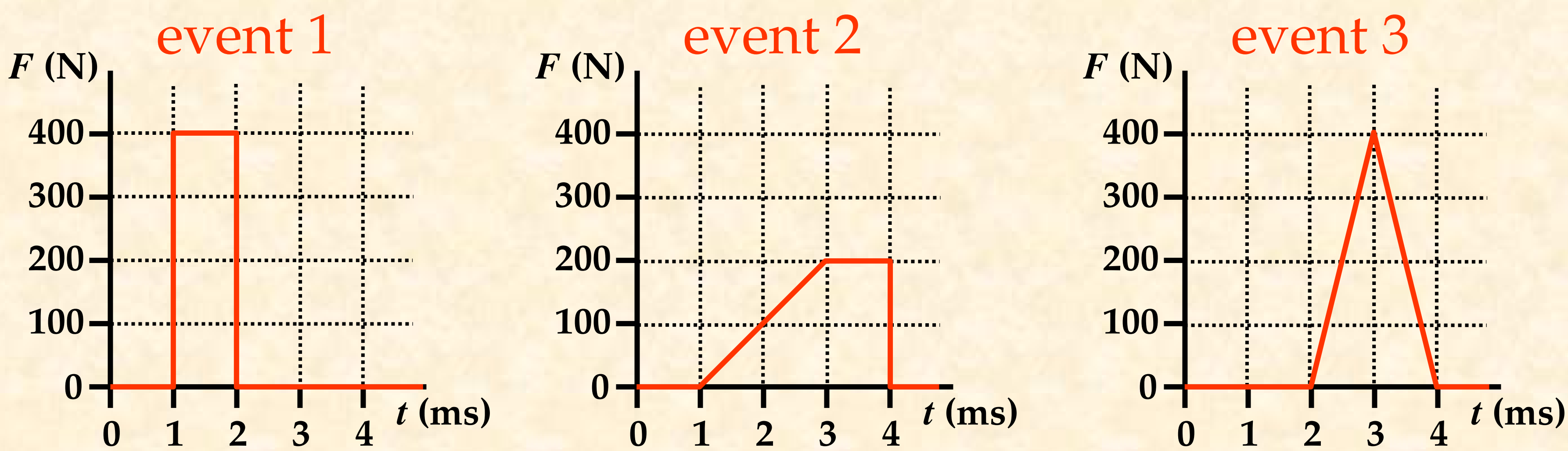


Which event changes the momentum of object B the most?

a) event 1 b) event 2 c) event 3 d) all the same

Chapter 9: Impulse and Momentum

Clicker question 9.3 In three separate interactions, object A exerts a time-dependent force over a relatively short time on object B, as represented in the $F(t)$ vs. t graphs below.



Which event changes the momentum of object B the most?

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Chapter 9: Impulse and Momentum

Newton's 2nd law and momentum:

$$\vec{F}_{\text{net}} = \sum \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

This is how Newton described his 2nd law in his ***Principia.***

- more general than $F = ma$ because it allows for the mass to change, such as in a rocket.

Impulse momentum theorem is just a restatement of Newton's 2nd Law. We use the I-M theorem when we don't care about the forces, accelerations, *etc.*, and need only a final velocity or momentum.

If we need an acceleration or a force, then we must use Newton's 2nd Law as we have been doing (*e.g.*, free-body diagrams, *etc.*)

9.3 Conservation of momentum

Consider a collision between two balls. What happens to their momenta?

$$\frac{d(p_x)_1}{dt} = F_{2\text{ on }1} \qquad \frac{d(p_x)_2}{dt} = F_{1\text{ on }2} = -F_{2\text{ on }1}$$

from Newton's 3rd Law. Thus,

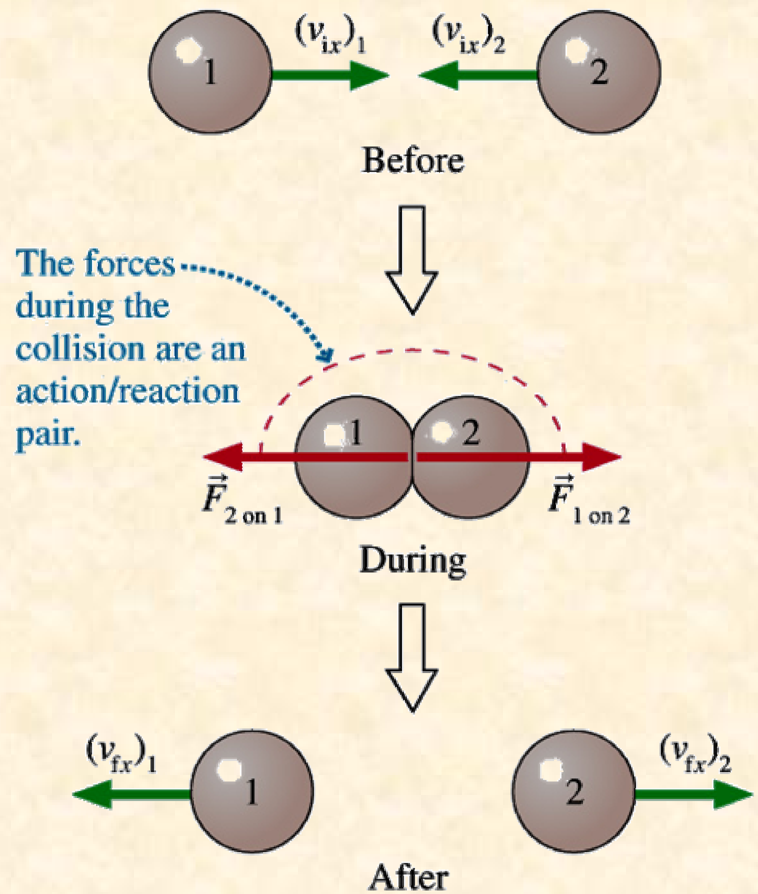
$$\frac{d(p_x)_1}{dt} + \frac{d(p_x)_2}{dt} = \frac{d}{dt}\left[(p_x)_1 + (p_x)_2\right] = 0$$

Anything whose time-derivative is zero is itself a constant. Thus,

$$(p_x)_1 + (p_x)_2 = \text{constant}$$

$$\Rightarrow \quad (p_{fx})_1 + (p_{fx})_2 = (p_{ix})_1 + (p_{ix})_2$$

and momentum is conserved through the collision.



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Chapter 9: Impulse and Momentum

Example: Find the velocity of the rail cars immediately after they collide and couple.

Before: $(p_{ix})_1 + (p_{ix})_2 = m_1(v_{ix})_1 + m_2(v_{ix})_2 = m_1 v_i$

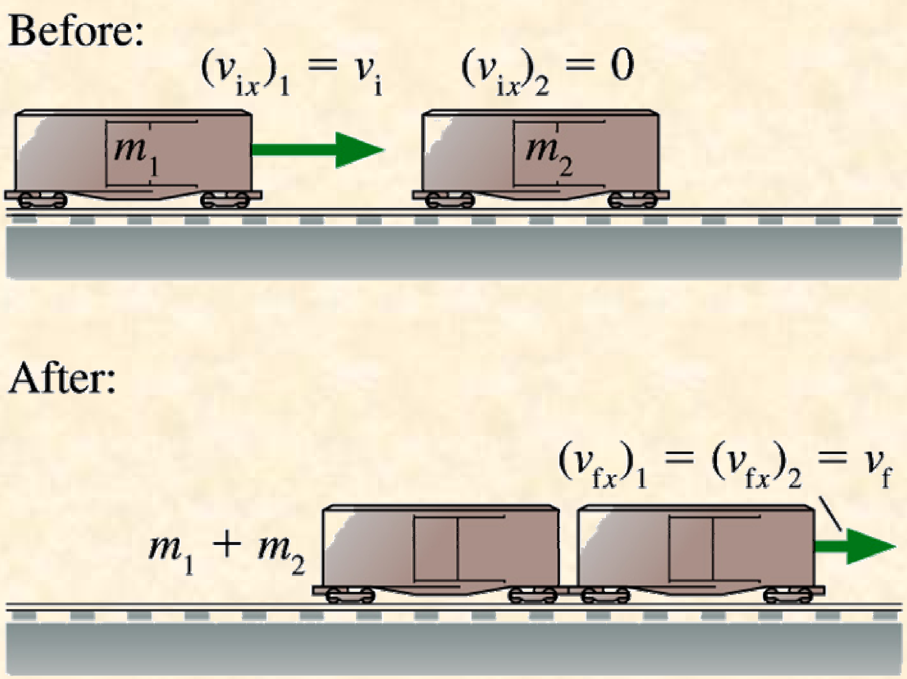
After: $(p_{fx})_1 + (p_{fx})_2 = m_1(v_{fx})_1 + m_2(v_{fx})_2 = (m_1 + m_2)\, v_f$

Conserve momentum:

$$(m_1 + m_2)\, v_f = m_1 v_i$$

$$\Rightarrow \quad v_f = \frac{m_1}{m_1 + m_2} v_i$$

$$\Rightarrow \quad v_f = \tfrac{1}{2} v_i \quad \text{for } m_1 = m_2.$$



Without knowing anything about the forces, we can find the final speed of the two cars.

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Conservation of momentum with many particles in 3-D

Consider a "system" with N particles.

Particles will experience *external* forces (coming from outside the system) and *internal* forces, all arranged in action/reaction pairs.

$$\text{Let } \vec{P} = \text{total momentum} = \sum_{k=1}^{N} \vec{p}_k$$

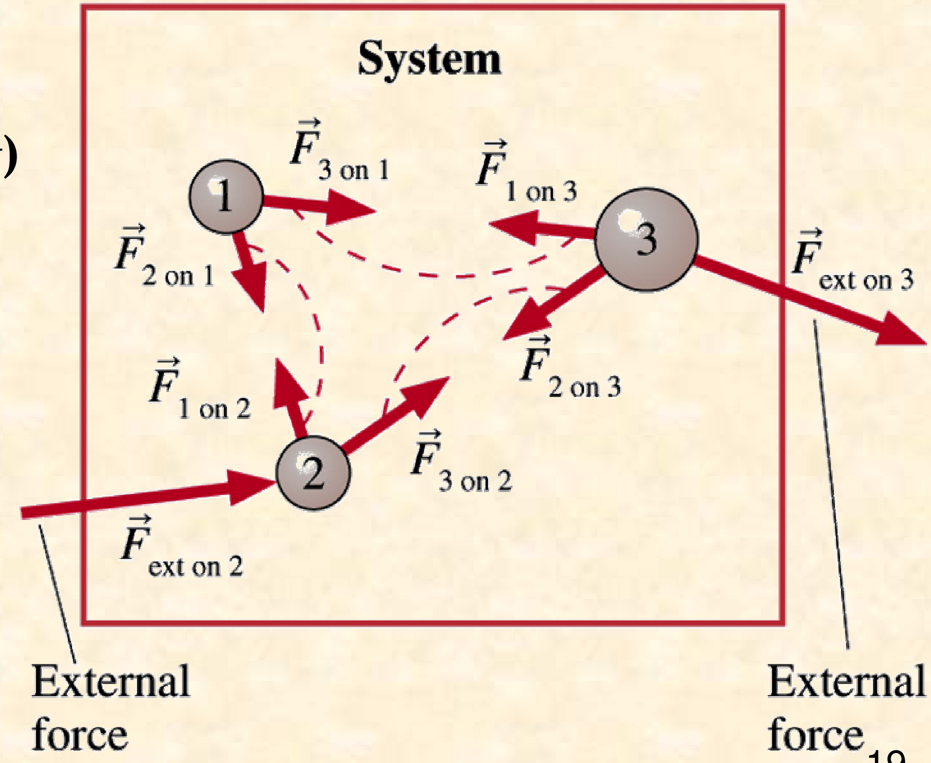
$$\frac{d\vec{P}}{dt} = \sum_{k=1}^{N} \frac{d\vec{p}_k}{dt} = \sum_{k=1}^{N} \vec{F}_k \quad \text{(Newton's 2}^{\text{nd}}\text{ Law)}$$

$$\text{Now, } F_k = \sum_{j \neq k}^{N} F_{j\text{ on }k} + F_{\text{ext on }k}$$

$$\frac{d\vec{P}}{dt} = \sum_{k=1}^{N} \vec{F}_k = \sum_{k=1}^{N} \sum_{j \neq k}^{N} \vec{F}_{j\text{ on }k} + \sum_{k=1}^{N} \vec{F}_{\text{ext on }k}$$

notes 9.1

$$\frac{d\vec{P}}{dt} = \sum_{k=1}^{N} \vec{F}_{\text{ext on }k} \quad \text{(Newton's 3}^{\text{rd}}\text{ Law)}$$



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***Definition*: An *isolated system* is one with no external forces acting on it.**

For many problems, having almost no external forces will also "count" as isolated.

Thus, for an isolated system, $\frac{d\vec{P}}{dt} = 0 \quad \Rightarrow \quad \boxed{\vec{P} = \text{constant}}$

Law of conservation of momentum: The total momentum, $\vec{P}$, of an isolated system is constant. Interactions within the system do not change the system's total momentum.

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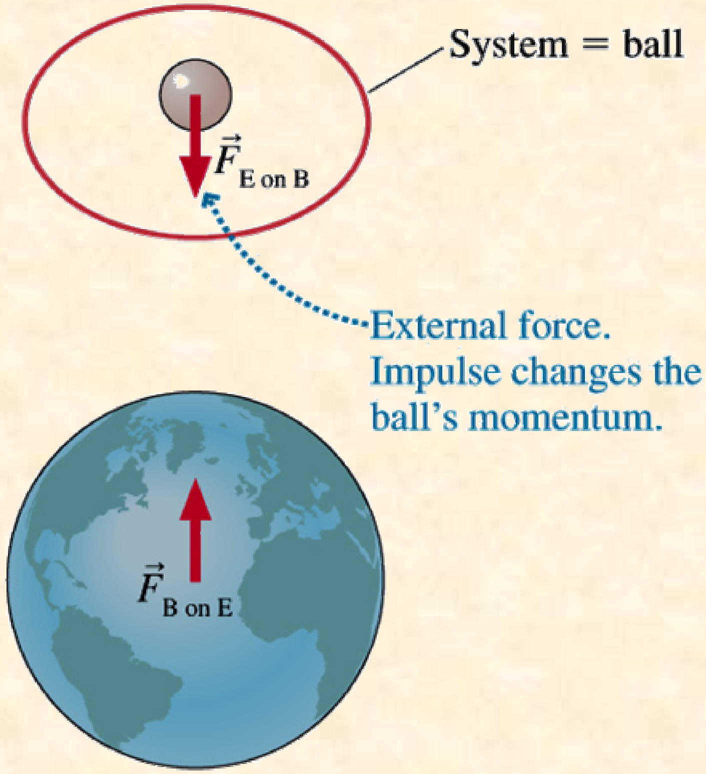
Choosing an appropriate system.

Drop a rubber ball on a hard floor. Is momentum of the ball conserved?

No. Ball picks up speed and momentum as it falls, then its momentum is reversed upon hitting the floor.

If we define our system as the ball, we wouldn't expect its momentum to be conserved, because is isn't isolated.

Earth's gravity and the normal force of the floor are both external forces to the ball, and these cause its momentum to change.



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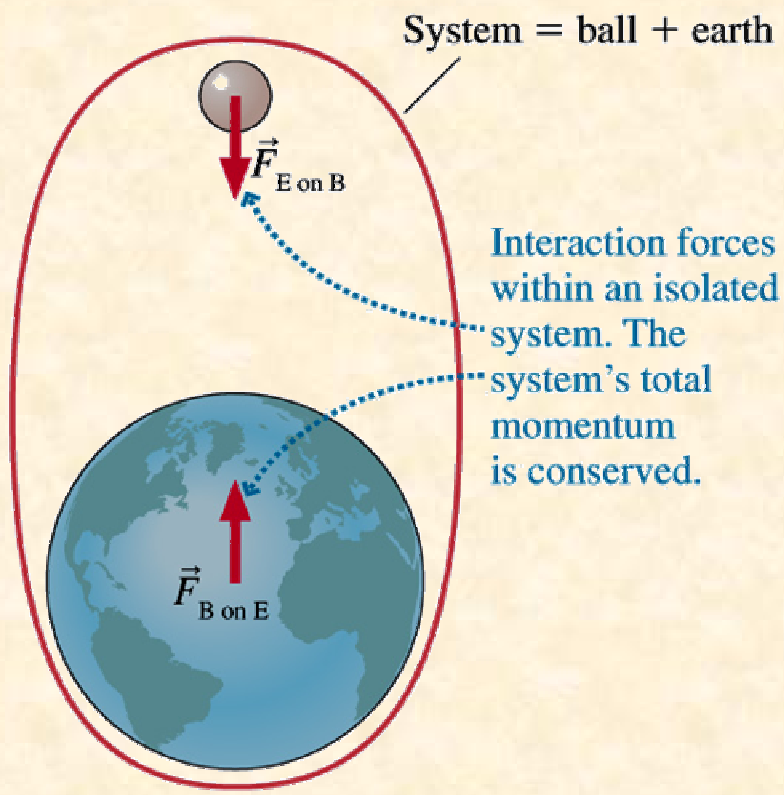
Now define the system as the ball + earth.

Both the gravitational force and collisional forces with the floor are now *internal* forces, and cannot change total momentum.

Thus, the total momentum of the ball + earth does not change as the ball falls, bounces, and rebounds.

If the ball is released from rest (relative to the earth), $P_i = 0$, and

$$P_f = m_b v_b + m_e v_e = P_i = 0 \quad \Rightarrow \quad v_e = -\frac{m_b}{m_e} v_b$$



for m_b = 60 g, v_b = –5 ms^{-1}, and m_e = 6 x 10^{24} kg, v_e = 5 x 10^{-26} ms^{-1} (would take the earth 300 million years to move the diameter of a single atom!)

Chapter 9: Impulse and Momentum

Overall strategy:

Try to choose your system so that all "complicated" forces are ***internal***. That way, the momentum of the system as a whole will not be affected by the forces you do not understand.

Try to arrange for your system to be isolated, so that there are no external forces, and momentum is conserved.

If it is not possible to isolate the system, break the problem up into segments so that during one of these segments, the system is isolated enough to use conservation of momentum.

Visualise: Draw "before" and "after" pictures, list known quantities, identify the quantities being sought.

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Example (explosion): **A projectile of mass m = 12 kg is launched at speed v_0 = 60 ms^{-1} and angle θ = 30^{o} relative to the horizontal. At the top of its trajectory, an explosion breaks the projectile into two pieces. The 8 kg piece is brought momentarily to rest, then drops straight down to the ground. How far from the launch site does the remaining 4 kg land?**

We break the problem into three parts:

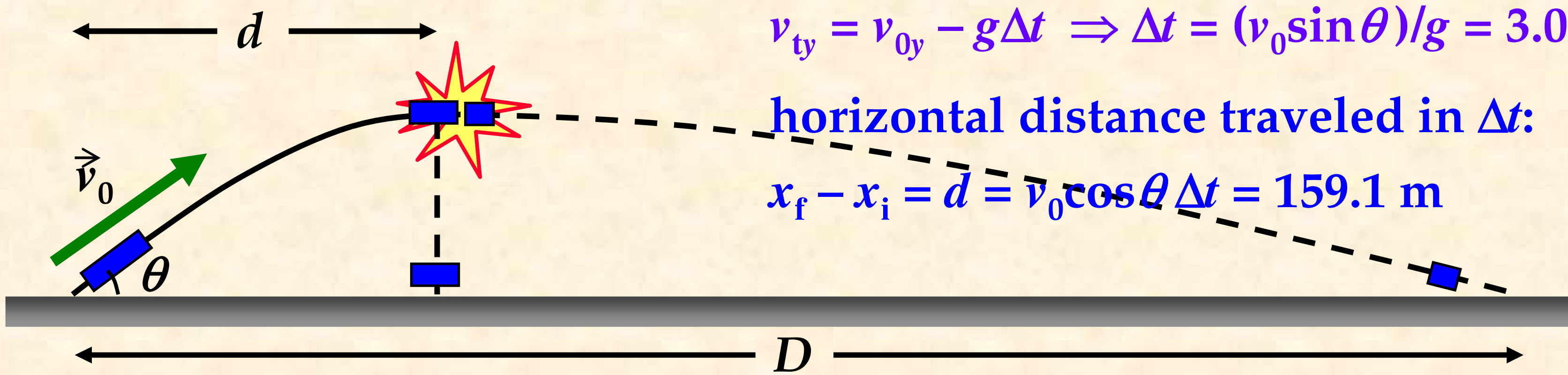
1. On the way up, the projectile remains intact. At the top of the trajectory, $v_{ty} = 0$ and $v_{tx} = v_{0x} = v_0\cos\theta$ = 52.0 ms^{-1}

time to get to top:

$v_{ty} = v_{0y} - g\Delta t \Rightarrow \Delta t = (v_0 \sin\theta)/g = 3.06$ s

horizontal distance traveled in Δt:

$x_f - x_i = d = v_0\cos\theta\,\Delta t = 159.1$ m



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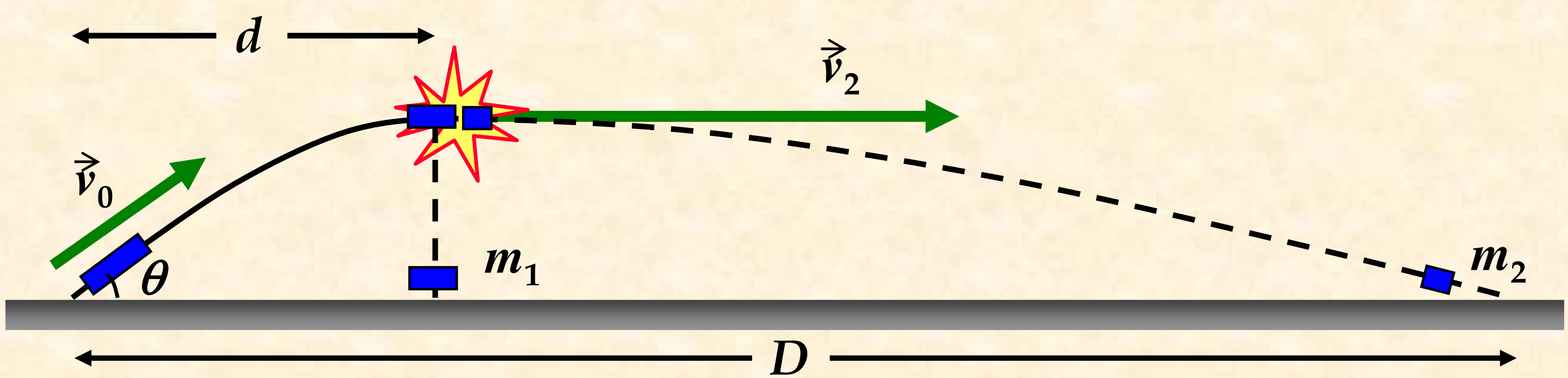
2. At the top, we conserve momentum immediately before and after the explosion. Let $m_1 = \frac{2}{3}m$ be the larger mass that drops to the ground, and let $m_2 = \frac{1}{3}m$ be the smaller mass that is propelled forward. Then,

$$\vec{p}_{\mathrm{i}} = m(v_{0x}, 0); \quad \vec{p}_1 = (0, 0); \quad \vec{p}_2 = \tfrac{1}{3}m\,(v_{2x}, v_{2y})$$

$$\text{Conserve momentum} \quad \Rightarrow \quad \vec{p}_{\mathrm{i}} = \vec{p}_1 + \vec{p}_2 \quad \Rightarrow \quad v_{2y} = 0 \ \text{ and } \ v_{2x} = 3v_{0x}$$

3. Use kinematics on the way down. It takes the same time to fall as it did to rise, and m_2 has three times the horizontal velocity m had. Thus, m_2 travels an additional $3d$ for a total distance, $D = 4d = 636$ m.



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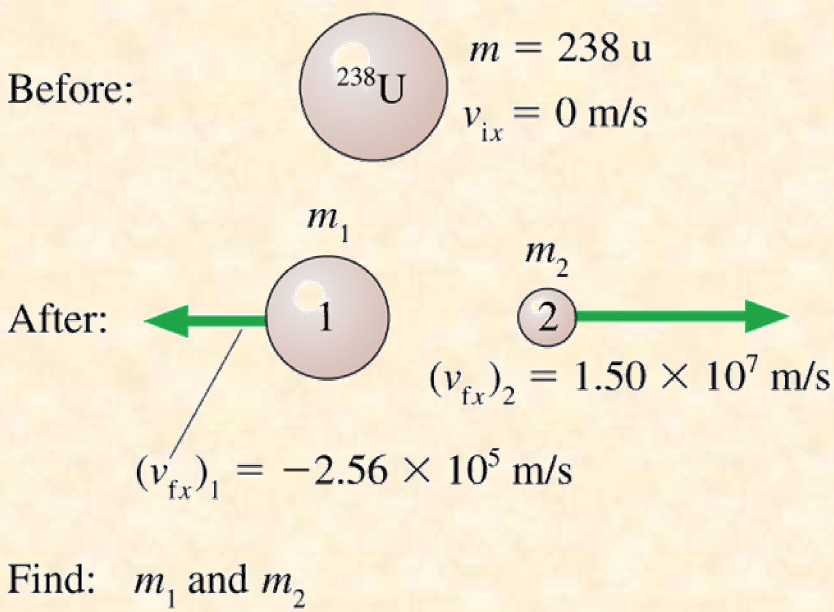
Example (explosion): **A ^{238}U (uranium) atom (with mass 238 "atomic units") spontaneously breaks up into a "daughter" nucleus of mass m_1 with a "recoil speed" of 2.56 x 10^5 ms^{-1}, and a high-energy particle of mass m_2 with a measured speed of 1.50 x 10^7 ms^{-1}. Find m_1 and m_2.**

$m_1 + m_2 = 238$ (conserve mass)



initial momentum:

$p_{ix} = 0$ (^{238}U is initially at rest)

final momentum:

$$p_{fx} = (p_{fx})_1 + (p_{fx})_2 = m_1(v_{fx})_1 + m_2(v_{fx})_2$$

conserve momentum:

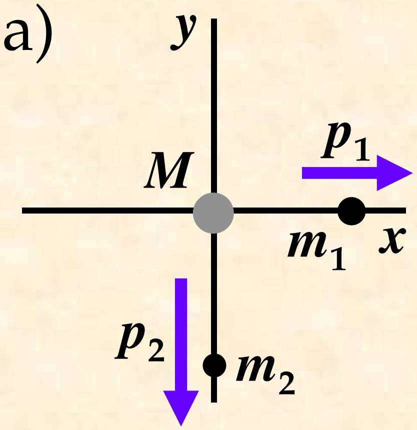
$$p_{fx} = p_{ix} \Rightarrow m_1(v_{fx})_1 + m_2(v_{fx})_2 = 0$$

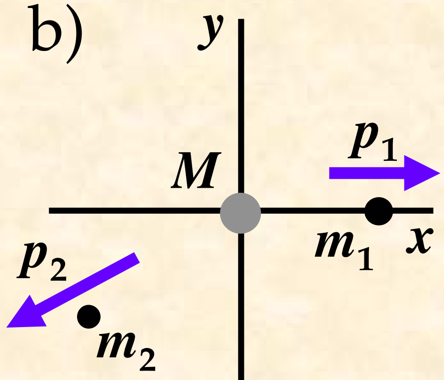
$$\Rightarrow m_2 = -m_1 \frac{(v_{fx})_1}{(v_{fx})_2} = 0.0171\, m_1 = 238 - m_1$$

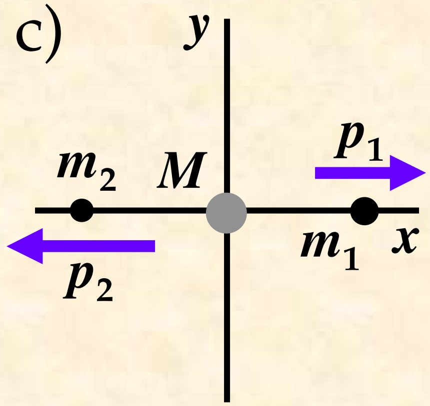
Solve for m_1: $m_1 = 234$ (thalium)

$\Rightarrow m_2 = 4$ (alpha particle, *i.e.*, helium)

Clicker question 9.4 An unstable nucleus of mass $\boldsymbol{M}$ is initially at rest and spontaneously decays into two smaller nuclei of masses $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ which speed off in two different directions. Which of the following is a ***plausible*** momentum diagram for the two daughter nuclei?

a) 

b) 

c) 

d) None; we need to know the masses first.

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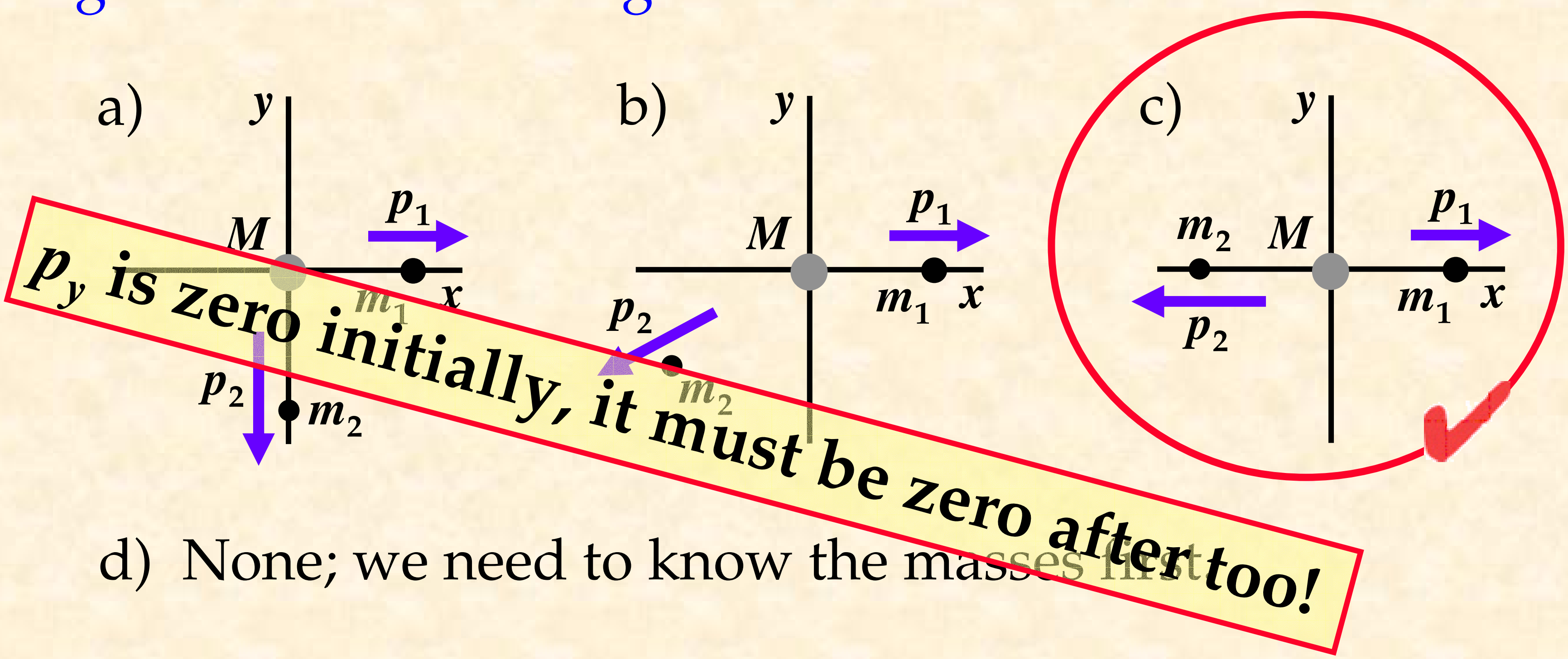

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Chapter 9: Impulse and Momentum

An ***inelastic collision*** is one where, after colliding, the two objects stick together and move off with a common velocity.

Example: Two masses, m_1 and m_2, approach each other as shown with $(v_{ix})_1$ known and $(v_{ix})_2$ unknown. After colliding, they move off together with a common velocity v_{fx}. Find $(v_{ix})_2$.

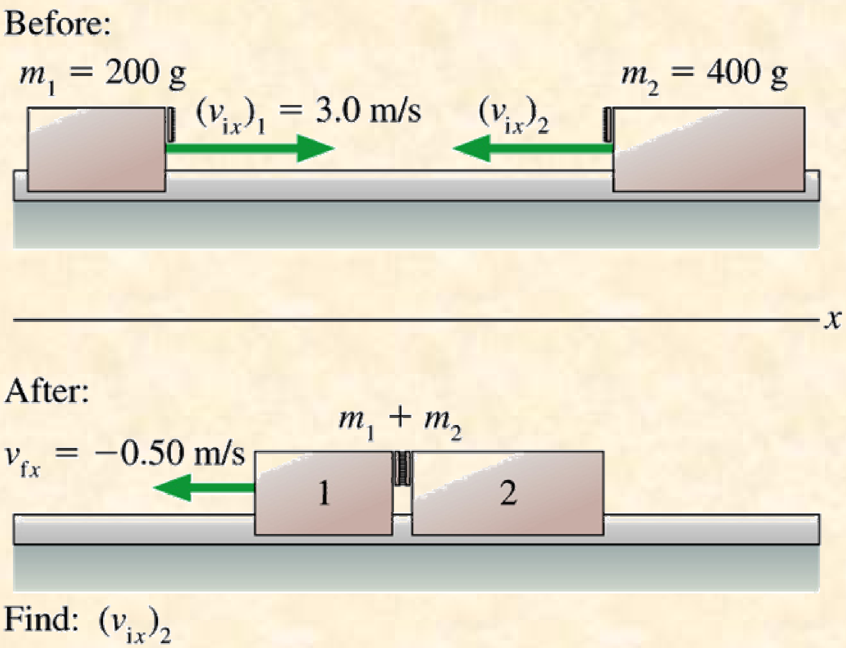
Conserve momentum:

$$p_f = p_i$$

$$\Rightarrow (m_1 + m_2)v_{fx} = m_1(v_{ix})_1 + m_2(v_{ix})_2$$

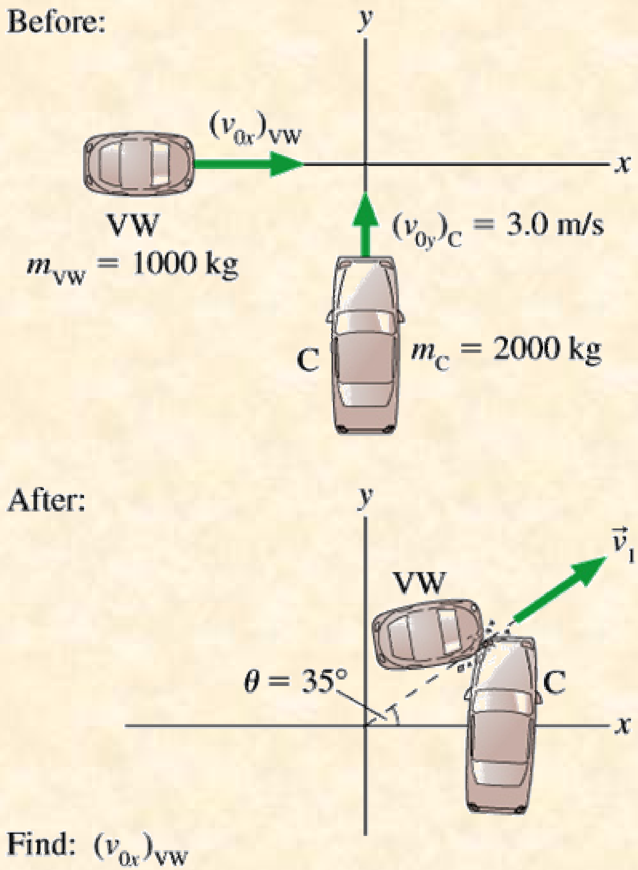
$$\Rightarrow (v_{ix})_2 = \frac{m_1 + m_2}{m_2} v_{fx} - \frac{m_1}{m_2}(v_{ix})_1$$

$$= -2.25 \text{ ms}^{-1}$$



Chapter 9: Impulse and Momentum

Example: Conservation of momentum in 2-D



In a car accident between a Cadillac (C) and a Volkswagen (VW), the two cars interlock bumpers and stay together after the collision. What was the velocity of the VW just before impact?

Conservation of momentum in its *vector* form: $\vec{p}_f = \vec{p}_i \Rightarrow p_{fx} = p_{ix}$ *and* $p_{fy} = p_{iy}$

$$p_{fx} = (m_C + m_{VW})v_1\cos\theta = m_{VW}(v_{0x})_{VW} = p_{ix}$$

$$p_{fy} = (m_C + m_{VW})v_1\sin\theta = m_C(v_{0y})_C = p_{iy}$$

Divide the red equation by the blue:

$$\frac{\cos\theta}{\sin\theta} = \frac{m_{VW}(v_{0x})_{VW}}{m_C(v_{0y})_C} \Rightarrow (v_{0x})_{VW} = (v_{0y})_C\frac{m_C}{m_{VW}}\cot\theta = \underline{8.6\ \mathrm{ms^{-1}}}$$