ASTR 3500

Assignment 4.

1. The Schwarzschild radius for a $10^8 M_\odot$ black hole is:

$$R_s = \frac{2GM}{c^2} = \frac{2 \times 6.6726 \times 10^{-11} \, Nm^2/kg^2 \times 10^8 \times 1.989 \times 10^{30} \, kg}{(2.9979 \times 10^8 \, m/s)^2}$$

$$= 2.9534 \times 10^{11} \, m$$

The density of the $10^8 M_\odot$ black hole is:

$$\rho = \frac{M}{\frac{4}{3}\pi R_s^3} = \frac{10^8 \times 1.989 \times 10^{30} \, kg}{4/3 \times \pi \times (2.9534 \times 10^{11} \, m)^3} = 1.8432 \times 10^3 \, kg/m^3$$

By comparison, the density of the Sun is:

$$\rho_\odot = \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3} = \frac{1.989 \times 10^{30} \, kg}{4/3 \times \pi \times (6.9598 \times 10^8 \, m)^3} = 1.4085 \times 10^3 \, kg/m^3$$

roughly 25% smaller than the density of the $10^8 M_\odot$ black hole.

The "surface gravity" for the $10^8 M_\odot$ black hole is:

$$g_{BH} = \frac{GM}{R_s^2} = \frac{6.6726 \times 10^{-11} \, Nm^2/kg^2 \times 10^8 \times 1.989 \times 10^{30} \, kg}{(2.9534 \times 10^{11} \, m)^2}$$

$$= 1.5215 \times 10^5 \, m/s$$

By comparison, the surface gravity of the Sun is:

$$g_\odot = \frac{GM_\odot}{R_\odot^2} = \frac{6.6726 \times 10^{-11} \, Nm^2/kg^2 \times 1.989 \times 10^{30} \, kg}{(6.9598 \times 10^8 \, m)^2}$$

$$= 2.7399 \times 10^2 \, m/s$$

almost 3 orders of magnitude smaller (actually ~555 times less)

2. The angular radius of an Einstein Ring is given by:

$$\theta_E = \sqrt{\frac{4GM}{c^2}\left(\frac{d_S - d_L}{d_S d_L}\right)} \text{ radians.}$$

We are given for MG 1654+1346 a value of: diameter = 2.1"

$$\theta_E = 2.1/2 \text{ arcseconds} = 1.05/206265 \text{ radian}$$

The distance of a distant galaxy from its redshift is given by:

$$d = \frac{c}{H_0}\left(\frac{(Z+1)^2-1}{(Z+1)^2+1}\right), \quad Z_S = 1.74 \text{ for source galaxy.}$$
$$Z_L = 0.25 \text{ for lensing galaxy.}$$

$$\therefore \text{for } Z_S, \ \frac{(Z_S+1)^2-1}{(Z_S+1)^2+1} = \frac{(1.74+1)^2-1}{(1.74+1)^2+1} = \frac{2.74^2-1}{2.74^2+1} = \frac{6.5076}{8.5076} = 0.764916$$

$$\text{for } Z_L, \ \frac{(Z_L+1)^2-1}{(Z_L+1)^2+1} = \frac{(0.25+1)^2-1}{(0.25+1)^2+1} = \frac{1.25^2-1}{1.25^2+1} = \frac{0.5625}{2.5625} = 0.2195122$$

$$\therefore d_S = 2.9979 \times 10^3 h^{-1} \times 0.764916 \text{ Mpc} = 2293.1\ h^{-1} \text{ Mpc}$$

$$d_L = 2.9979 \times 10^3 h^{-1} \times 0.2195122 \text{ Mpc} = 658.08\ h^{-1} \text{ Mpc}$$

$$\therefore M_L = \frac{c^2}{4G}\left(\frac{d_S d_L}{d_S - d_L}\right)\theta_E^2 \quad \text{in proper units, } c \text{ in m/s, } d \text{ in m}$$

$$= \frac{(2.9979 \times 10^8 \text{ m/s})^2}{4 \times 6.6726 \times 10^{-11} \text{ Nm}^2/\text{kg}^2} \cdot \left(\frac{1.05}{206265}\right)^2 h^{-1}\left(\frac{658.08 \times 2293.1}{2293.1 - 658.08}\right) 3.0857 \times 10^{22} \text{ m/Mpc}$$

$$= 3.3673 \times 10^{26} \text{ kg/m} \times 2.5914 \times 10^{-11} h^{-1} \times 9.2295 \times 10^2 \times 3.0857 \times 10^{22} \text{ m}$$

$$= 2.4851 \times 10^{41} \text{ kg } h^{-1}$$

$$= \frac{2.4851 \times 10^{41} \text{ kg } h^{-1}}{1.989 \times 10^{30} \text{ kg}/M_\odot}$$

$$= 1.2494 \times 10^{11} h^{-1} M_\odot$$

For $h = 0.65$, $H_0 = 65$ km/s/Mpc.

$M_L = 1.9 \times 10^{11} M_\odot$, fairly typical of a large spiral galaxy or elliptical galaxy.

3. For the average density of baryonic matter we assume a value of 4% of the critical density, given as $4.17 \times 10^{-28}\ \mathrm{kg/m^3}$ in the textbook (p. 1150).

The equivalent mass density of photons comes from the energy density of black-body radiation, $u_{rad} = aT_{bb}^4$

If the energy comes from the conversion of baryonic density into radiation, then $E = \rho_{b,0}c^2$, where $\rho_{b,0}$ is the current density of baryonic matter,

i.e. $\rho_{b,0}c^2 = aT_{bb}^4$, so

$$T_{bb} = \left(\frac{\rho_{b,0}c^2}{a}\right)^{1/4} = \left[\frac{4.17\times10^{-28}\ \mathrm{kg/m^3}\,(2.9979\times10^{8}\ \mathrm{m/s})^2}{(7.565767\times10^{-16}\ \mathrm{J/m^3K^4})}\right]^{1/4}$$

$$= (4.9536\times10^{4}\ \mathrm{K^4})^{1/4}$$

$$= 14.92\ \mathrm{K}$$

According to Wien's Law, the wavelength at which the radiation peaks is given by: $\lambda_{max} = (0.00290\ \mathrm{m\,K})/T$

For $T_{bb} = 14.92\,\mathrm{K}$, $\lambda_{max} = \dfrac{2.9\ \mathrm{mm}}{14.92} = 0.19\ \mathrm{mm}$, in the far infrared section of the electromagnetic spectrum.

There is too little matter in the universe to cause the sky to glow if it were converted entirely into radiation.

4. For a redshift of $z = 1.776$, the scale factor is given by: $R = \dfrac{1}{1+z} = \dfrac{1}{1+1.776} = \dfrac{1}{2.776} = 0.3602305$

According to equation 29.58, the blackbody temperature of the cosmic microwave background radiation varies as:

$$RT = T_0.$$

Here, $T_0 = 2.726$ K

So, $T(\text{at } z = 1.776) = \dfrac{T_0}{R}$

$$= \frac{2.726\text{ K}}{0.3602305}$$

$$= 7.57 \text{ K}$$

The value is in close agreement with the measured temperature of the cloud of 7.4 ± 0.8 K, the two values agreeing to within the uncertainty of the latter value.

It appears that the CMB radiation can account entirely for the heating of the cloud.

5. The equation to be solved is:

$$\frac{m_H R^3}{f\rho_{b,0}}\left[\frac{2\pi m_e k T_0}{h^2 R}\right]^{3/2} e^{-\chi_I R/kT_0} = \frac{f}{1-f}$$

We wish to solve the equation for the case $f = 0.5$, i.e half of the electrons and protons have combined as neutral atoms.

We also have:

$$\begin{aligned}
\rho_{b,0} &= 4.17\times10^{-28}\ \mathrm{kg/m^3}\\
m_H &= 1.673532499\times10^{-27}\ \mathrm{kg}\\
m_e &= 9.10938188\times10^{-31}\ \mathrm{kg}\\
h &= 6.62606876\times10^{-34}\ \mathrm{J.s}\\
k &= 1.3806503\times10^{-23}\ \mathrm{J/K} = 8.6173423\times10^{-5}\ \mathrm{eV/K}\\
\chi_I &= 13.6\ \mathrm{eV}\ \text{for hydrogen.}\\
T_0 &= 2.726\ \mathrm{K}
\end{aligned}$$

Thus, we want $\frac{f}{1-f} = \frac{0.5}{1-0.5} = 1 = AR^{1.5}Be^{-CR}$

where $A = \frac{m_H}{0.5\rho_{b,0}} = \frac{1.673532499\times10^{-27}}{0.5\times4.17\times10^{-28}} = 8.0265$

$$B = \left(\frac{2\pi m_e k T_0}{h^2}\right)^{3/2} = \left[\frac{2\pi\times9.10938\times10^{-31}\times1.3807\times10^{-23}\times2.726}{(6.6261\times10^{-34})^2}\right]^{3/2}$$

$$= 1.0868\times10^{22}$$

$\therefore AB = 8.0265\times1.0868\times10^{22} = 8.7232\times10^{22}$

$$C = \frac{\chi_I}{kT_0} = \frac{13.6}{8.617\times10^{-5}\times2.726} = 5.7895\times10^4$$

Set up as a trial and error test in Excel, the solution rapidly converges to $R = 7.25088\times10^{-4}$ for the scale factor

The temperature follows from $T = T_0/R$

$$= 2.726\ \mathrm{K}/7.25088\times10^{-4}$$

$$= 3783\ \mathrm{K}$$

R	8.72E+22 -5.79E+04	e-term	product
1		0.000000000000000000	0.0000000000000000
0.1		0.0000000000000000	0.0000000000000000
0.01		0.0000000000000000000000000	0.0000000000000000
1.0E-03		0.0000000000000000000000001	0.0000001982425156
0.0001		0.0030595115498230100000000	266887311514161
0.0002		0.0000093606109235003700000	2309537495054
0.0003		0.0000000286388972338489000	12981174948
0.0004		0.0000000000876210368611545	61146866
0.0005		0.0000000000002680775742842	261452
0.0006		0.0000000000000008201864348	1052
0.0007		0.0000000000000000025093699	4.05404
0.0008		0.0000000000000000000076774	0.01515
0.00071		0.0000000000000000014064681	2.32110
0.00072		0.0000000000000000007883065	1.32852
0.00073		0.0000000000000000004418352	0.76019
0.000725		0.0000000000000000005901708	1.00499
0.000726		0.0000000000000000005569731	0.95042
0.0007251		0.0000000000000000005867639	0.99939
0.000725088		0.0000000000000000005871716	1.00006