## Tensor Calculus

Before considering how we ever define differentiation on a manifold, lets look at whether the partial derivative of a tensor is a tensor:

$$\partial'_c X'^a = \frac{\partial}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^b} X^b\right)$$

$$= \frac{\partial x^d}{\partial x'^c}\frac{\partial}{\partial x^d}\left(\frac{\partial x'^a}{\partial x^b} X^b\right)$$

$$= \frac{\partial x^d}{\partial x'^c}\frac{\partial^2 x'^a}{\partial x^d \partial x^b} X^b + \frac{\partial x^d}{\partial x'^c}\frac{\partial x'^a}{\partial x^b}\frac{\partial X^b}{\partial x^d}$$

$$= \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\partial_d X^b + \frac{\partial^2 x'^a}{\partial x^d \partial x^b}\frac{\partial x^d}{\partial x'^c} X^b$$

This is clearly not a tensor as the second term doesn't transform in a tensor manner. (HINT for thought: What if we could take that term out or cancel it somehow?)

Recall from basic calculus that the derivative is a limiting process of two function evaluations at <u>different</u> points

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The analogous definition for tensors would be

$$\lim_{\delta u \to 0} \frac{[X^a]_P - [X^a]_Q}{\delta u}$$

where $\delta u$ defines the separation.

This is problematic since the coordinate transforms at P & Q will in general be different, ie.

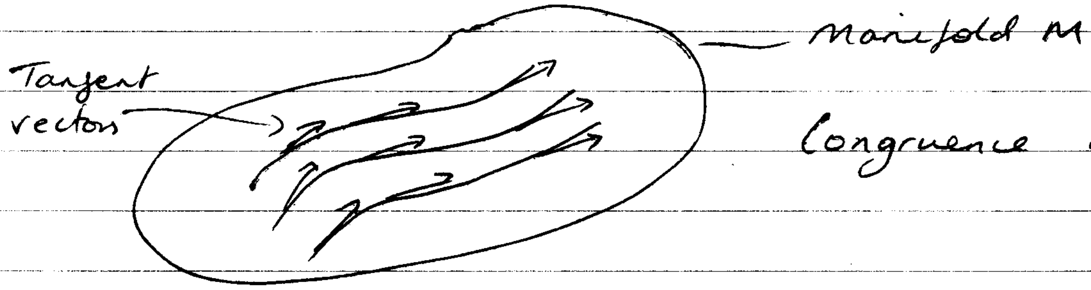
$$X'^{a}_{P} = \left[\frac{\partial x'^{a}}{\partial x^{b}}\right]_{P} X^{b}_{P} \quad \text{and similarly at } Q.$$

So to define tensorial derivatives we need think carefully about howe we "drag" a tensor at one place to another.

The Lie Derivative

We need to first describe how coordinates change on the manifold itself. This is known as an active coordinate change.

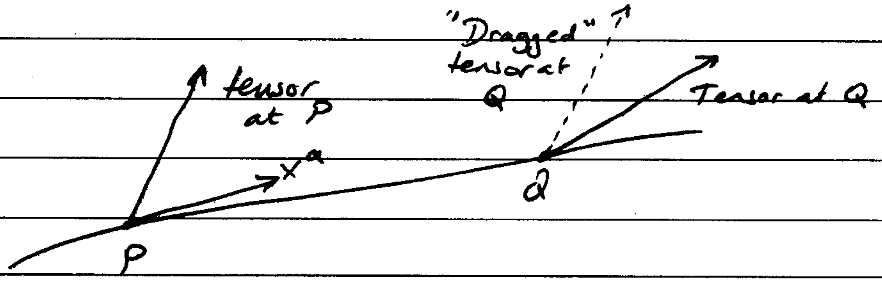
To define the coordinate change we first introduce the idea of a "congruence of curves," that connect points on the manifold together.




You can think of these curves as streamlines on the manifold. Given a vector field we find the curves by solving

$$\frac{dx^a}{du} = X^a(x(u))$$

We will use these curves to "drag" a tensor at one position $P$, to another at $Q$.




Note: this necessarily implies we are considering tensor fields, ie $T^{a\ldots}_{b\ldots}(P)$ would be the value of the tensor field at $P$.

Consider:

$$x^a \longrightarrow x'^a = x^a + \delta u\, X^a(x)$$

By definition, we will take this to define the shift from position $P$ to $Q$, ie $\overrightarrow{PQ}$.

Then $$\frac{\partial x'^a}{\partial x^b} = \delta^a_b + \delta u\, \partial_b X^a$$

Under the process of "dragging" (using a rank 2 contravariant tensor for simplicity)

$$T^{ab}(x) \longrightarrow T'^{ab}(x')$$

We also have a second tensor at $Q$, that give by the tensor found from the active coordinate change, $T^{ab}(x')$. ($T'^{ab}(x')$ and $T^{ab}(x')$ are <u>not</u> the same.)

Since $T'^{ab}(x')$ is a tensor

$$T'^{ab}(x') = \frac{\partial x'^a}{\partial x^c}\frac{\partial x'^b}{\partial x^d} T^{cd}(x)$$

$$= (\delta^a_c + \delta u\, \partial_c X^a)(\delta^b_d + \delta u\, \partial_d X^b)\, T^{cd}(x)$$

$$(A) \quad = T^{ab}(x) + \left[\partial_c X^a T^{cb}(x) + \partial_d X^b T^{ad}(x)\right]\delta u + O(\delta u^2)$$

For the tensor found by the active coordinate change we apply Taylor expansion

$$T^{ab}(x') = T^{ab}(x^c + \delta u\, X^c(x)) = T^{ab}(x) + \delta u\, X^c \partial_c T^{a}$$
$$= (B)$$

The Lie derivative is defined as:

$$L_X T^{ab} = \lim_{\delta u \to 0} \frac{T^{ab}(x') - T'^{ab}(x')}{\delta u}$$

Note how it depends on $X^a$

$$= X^c \partial_c T^{ab} - T^{ac}\partial_c X^b - T^{cb}\partial_c X^a$$

(subtract (A) from (B)).

Properties of the Lie Derivative:

Linear: $L_X(\lambda Y^a + \mu Z^a) = \lambda L_X Y^a + \mu L_X Z^a$
where $\lambda, \mu$ are reals.

Leibniz: $L_X(Y^a Z_{bc}) = Y^a (L_X Z_{bc}) + (L_X Y^a) Z_{bc}$

Commutes with contraction: $\delta^a_b L_X T^a{}_b = L_X T^a{}_a$

Lie derivative of scalar field: $L_X \phi = X\phi = X^a \partial_a \phi$

Lie derivative of a contravariant vector field: $L_X Y^a = [X, Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a$

Lie derivative of a covariant vector field $L_X Y_a = X^b \partial_b Y_a + Y_b \partial_a X^b$

Type preserving: The Lie derivative of a type $(r,s)$ tensor is another type $(r,s)$ tensor.

See P. 72 for the definition of the Lie derivative of an arbitrary type $(r,s)$ tensor.

For all our effort, the Lie derivative is not quite what we want. The type preserving nature means it is not precisely analogous to $\partial_a$ (although coordinates can be chosen to make it appear that way).

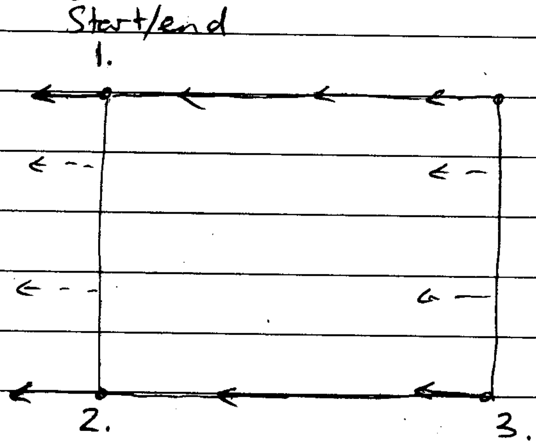
Lie derivatives are extremely useful though for defining a special class of transformations, called isometries, which have the property that ~~spatial~~ distance relationships are unchanged following the ~~transformation~~ transformation.

The covariant derivative will be shown to have the properties we want.

Note: Prior to Einstein using Riemannian geometry, all physics used partial derivatives!

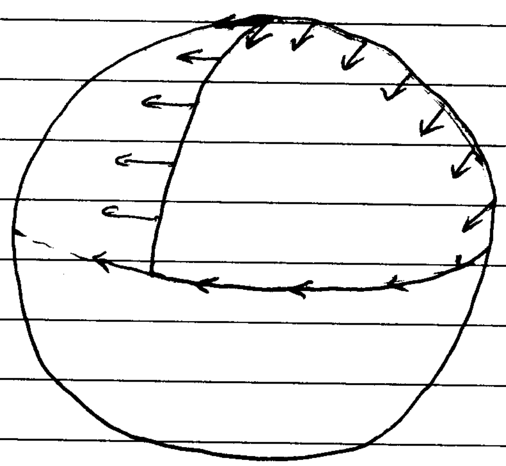
# Concept behind "Parallel Transport"

Consider standing on a 2-d plane with your arm outstretched. Walk forward 5 paces, sidestep to the left 5 paces, walk backwards 5 paces and lastly sidestep to the right 5 paces. What happens to the vector corresponding to where your arm points?




Points in the same direction at the final destination.

If we perform the same exercise on the surface of the Earth, start from the N. Pole → equator → to second point on equator → back to N. Pole.



The starting & ending vectors no longer line-up. The transport process was the same though. The change is caused by the basis vectors on the sphere changing from point-to-point

In the definition of the Lie derivative we used a congruence of curves to transport a tensor, however alternative definitions are possible.

Consider two points $P \equiv P(x)$ and $Q \equiv Q(x+\delta x)$. We seek to define a precise one-to-one correspondence between vectors at P & Q. (really a correspondence between ~~tangent~~ tangent spaces $T_P(M)$ & $T_Q(M)$). For reasons that will become clear, we'll define the corresponding vectors as parallel.

What requirements can we place on the map?
(1) When $P \equiv Q$ the map must reduce to the identity
(2) The simplest transformation law we can consider is linear, and this will give a one-to-one map

Suppose we denote a vector at P by $X^a(x)$ then we define the transported vector at Q to be

$$X^a(x) + \bar{\delta} X^a(x)$$

The linearity requirement means that we can write

$$\bar{\delta} X^a = W^a{}_b X^b$$

where the $W^a{}_b$ must depend on P & Q.

We can satisfy the requirements (1) and (2) by setting

$$W^a{}_b = \delta^a_b - \Gamma^a{}_{bc}\, \delta x^c$$

Clearly
$$\lim_{\delta x \to 0} W^a{}_b = \delta^a_b$$
and we recover the identity map when $P \equiv Q$.

Thus for the transported vector at $Q$
$$X^a(x) + \bar{\delta} X^a(x) = (\delta^a_b - \Gamma^a{}_{bc}\,\delta x^c)\,X^b(x)$$
$$= X^a - \Gamma^a{}_{bc}\,\delta x^c\,X^b(x)$$

the $\Gamma^a{}_{bc}$ must depend only on $P$ (the separation to $Q$ comes in through the $\delta x^c$). For an $n$-dimensional manifold $\Gamma^a{}_{bc}$ corresponds to $n^3$ functions. Note the minus sign is necessary for convention.

The $\Gamma^a{}_{bc}$ are called <u>connection coefficients</u>.

To define a derivative we also need to consider the vector field at $Q$. By Taylor expansion:
$$X^a(x+\delta x) = X^a(x) + \delta x^b \partial_b X^a$$

Given $X^a(x+\delta x)$ and the transported vector $X^a(x) + \bar{\delta} X^a(x)$ we now define the covariant derivative:
$$\nabla_c X^a = \lim_{\delta x \to 0} \frac{1}{\delta x^c}\left\{ X^a(x+\delta x) - (X^a(x) + \bar{\delta} X^a(x))\right\}$$
Since they are both vectors at $Q$ this is an acceptable tensorial definition.

Notation: The covariant derivative of a vector $X^a$ is also written $X^a{}_{;c}$ and $X^a{}_{||c}$.

In the limit $\delta x \to 0$

$$\nabla_c X^a = \partial_c X^a + \Gamma^a{}_{bc} X^b$$

note that the vectors sum into the connection on the first covariant index, while the differentiation index, c, is the last covariant index.

Although we've set up the derivative in terms of two vectors at Q, the requirement that $\nabla_c X^a$ be tensor will constrain how the $\Gamma^a{}_{bc}$ transform.

Writing $$\nabla'_c X'^a = \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\left\{ \partial_d X^b + \Gamma^b{}_{ed} X^e \right\}$$
$$= \partial'_c X'^a + \Gamma'^a{}_{bc} X'^b$$

We can expand out $\partial'_c X'^a$ to derive an equation for the $\Gamma'^a{}_{bc}$ (exercise)

$$\Gamma'^a{}_{bc} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}\Gamma^d{}_{ef} - \frac{\partial x^d}{\partial x'^b}\frac{\partial x^e}{\partial x'^c}\frac{\partial^2 x'^a}{\partial x^d \partial x^e}$$

Equivalently since $\partial_d \left\{ \frac{\partial x^a}{\partial x'^c}\frac{\partial x'^c}{\partial x^b} \right\} = \partial_d \left\{ \delta^a_b \right\} = 0$

$$\Gamma'^a{}_{bc} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}\Gamma^d{}_{ef} + \frac{\partial x'^a}{\partial x^d}\frac{\partial^2 x^d}{\partial x'^b \partial x'^c}$$

Any set of functions that transform according to this transformation law is called an <u>affine connection</u>.

Notice how the second term plays the role of compensating for the additional components associated with the partial derivative under coordinate changes.

## <u>Properties of Covariant Derivatives</u>

(1) $\nabla_a \phi = \partial_a \phi$ where $\phi$ is a scalar

(2) $\nabla_a$ can be made to be Liebniz by setting

$$\nabla_a X_b = \partial_a X_b - \Gamma^c{}_{ba} X_c$$

Notice that the connection coefficients come in with a <u>minus sign</u> when differentiating a covariant vector.

(3) For a general tensor

$$\nabla_c T^{a\ldots}_{b\ldots} = \partial_c T^{a\ldots}_{b\ldots} + \Gamma^a{}_{dc} T^{d\ldots}_{b\ldots} + \ldots - \Gamma^d{}_{bc} T^{a\ldots}_{d\ldots} - \ldots$$

Terms sum over the upper indices contracted into the $1^{st}$ lower index in $\Gamma^a{}_{bc}$, and subtract over the lower indices contracted into the upper index of $\Gamma^a{}_{bc}$.