Black Holes Don't Suck.

We've shown, despite what popular science fiction might lead you to believe, that black holes do not exert any more gravitational pull often a star of mass M (provided we aren't inside the event horizon of course). There is also a single special stable orbit in the Schwarzschild metric at $r = 3m$.

One of the critical issues in the formation of supermassive black holes (masses $> 10^5 M_\odot$ which reside in the centre of galaxies) is how the material actually falls down to the Schwarzschild radius. This requires vast amounts of angular momentum to be shed by infalling matter, and is the subject of ongoing research.

However, there is one sense in which black holes are harder to get away from than their Newtonian counterpart. If we set up a rocket ship with enough thrust to counter the pull of the black hole, the four force required to keep the position balanced is

$$f^\alpha = m\left\{\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right\}$$

As it turns out, in the observer's frame, the required force is

$$f^{\hat{r}} = m\left(1 - \frac{2M}{r}\right)^{-1/2} \frac{M}{r^2}$$

which becomes infinite as $r \to 2m$, in agreement with the idea that the event horizon is a point of no return.

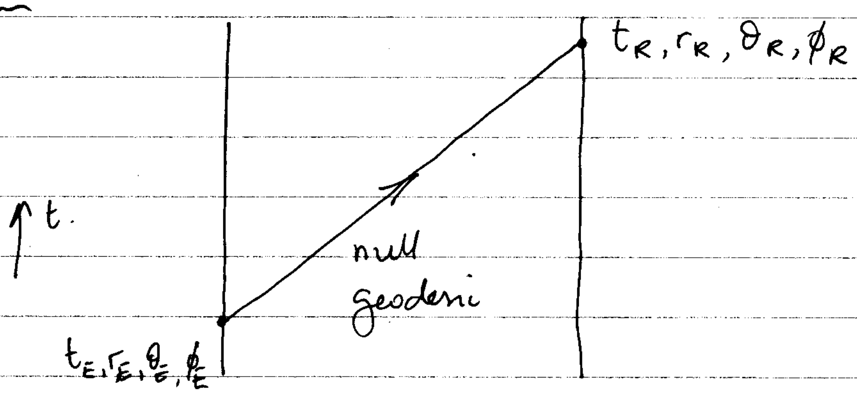
Spectral Shift (or Gravitational Redshift)

Thus far we have not considered what happens to photons leaving the vicinity of a black hole (or other massive object). ~~If~~ We define the photons to be emitted from a point (r_E, θ_E, ϕ_E) and received at a point (r_R, θ_R, ϕ_R). For now, ~~using~~ the full coordinates of emission & reception are $(t_E, r_E, \theta_E, \phi_E)$ and $(t_R, r_R, \theta_R, \phi_R)$. Using an affine parameter u to parameterize the geodesic ($u = u_E$ at emission, $u = u_R$ at reception) the line element is

$$0 = c^2\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin\theta^2\dot{\phi}^2\right)$$

$$\Rightarrow \quad c^2\left(1 - \frac{2m}{r}\right)\dot{t}^2 = \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$

and we can represent the events via a space-time diagram



Returning to the line element, we could also write

$$c^2\left(1-\frac{2m}{r}\right)\dot{t}^2 = -g_{ij}\,\dot{x}^i\dot{x}^j$$

and hence

$$\dot{t} = \frac{1}{c}\left[\left(1-\frac{2m}{r}\right)^{-1}\tilde{g}_{ij}\,\dot{x}^i\dot{x}^j\right]^{\frac{1}{2}}$$

where $\tilde{g}_{ij} = -g_{ij}$

Integrating gives

$$t_R - t_E = \frac{1}{c}\int_{u_E}^{u_R}\left[\left(1-\frac{2m}{r}\right)^{-1}\tilde{g}_{ij}\,\dot{x}^i\dot{x}^j\right]^{\frac{1}{2}}du$$

As long as we require that the spacetime is static (ie no t dependence in the $\tilde{g}_{ij}$) the the RHS depends only on the path through the spacetime. Thus at any time, photons sent from the emitter to the receiver will ~~be~~ have a fixed $t_R - t_E$. Hence for a pair of signals, 1 & 2,

$$t_R^{(1)} - t_E^{(1)} = t_R^{(2)} - t_E^{(2)}$$

which can be rearranged to give

$$\Delta t_R \equiv t_R^{(2)} - t_R^{(1)} = t_E^{(2)} - t_E^{(1)} \equiv \Delta t_E$$

So the coordinate time difference between receptions & emissions are the same.

However, as we have emphasized before, coordinate time is not measured by observers in the Schwarzschild geometry. We must use proper time.

For the emission point at constant r, θ, ϕ the proper time is given by

$$c^2 d\tau^2 = c^2\left(1 - \frac{2m}{r}\right) dt^2$$

hence for small time differences Δt we can write

$$\Delta\tau = \left(1 - \frac{2m}{r}\right)^{1/2} \Delta t$$

this will be true for observers at rest at both the emission & reception points. Hence

$$\Delta\tau_E = \left(1 - \frac{2m}{r_E}\right)^{1/2} \Delta t_E$$

$$\Delta\tau_R = \left(1 - \frac{2m}{r_R}\right)^{1/2} \Delta t_R$$

Using the equivalence of $\Delta t_R = \Delta t_E$, we get

$$\Delta\tau_E = \left(1 - \frac{2m}{r_E}\right)^{1/2} \Delta\tau_R \left(1 - \frac{2m}{r_R}\right)^{-1/2}$$

and hence

$$\frac{\Delta\tau_R}{\Delta\tau_E} = \left(\frac{1 - \frac{2m}{r_R}}{1 - \frac{2m}{r_E}}\right)^{1/2} \qquad \text{———} (1)$$

We can now use this formula to see what happens to the emitted photons. Suppose n wavecycles are emitted in a time $\delta\tau_E$. The net frequency of the radiation will be

$$\nu_E = \frac{n}{\Delta\tau_E}$$

There will of course by n cycles received, with a time separation between beginning & end of $\Delta\tau_R$. The net frequency of the received radiation is then

$$\nu_R = \frac{n}{\Delta\tau_R}$$

Clearly, since $\Delta\tau_R \neq \Delta\tau_E$ the frequencies will be different. Since $\nu \propto \frac{1}{\Delta\tau}$, using (1) we find

$$\frac{\nu_R}{\nu_E} = \left(\frac{1-\frac{2m}{r_E}}{1-\frac{2m}{r_R}}\right)^{1/2} = \left(\frac{1-\frac{2GM}{r_E c^2}}{1-\frac{2GM}{r_R c^2}}\right)^{1/2}$$

Hence, assuming $r_R > r_E$, we get $\nu_R < \nu_E$ and the frequency has been lowered, ie red-shifted. Swapping the position of r_E & r_R will lead to a blue shift.

If $r_E c^2 \gg GM$ (ie a long way from the gravitational source) and $r_R c^2 \gg GM$ as well we can expand the square root using Taylor series to get

$$\frac{\nu_R}{\nu_E} \simeq \left(1 - \frac{1}{2}\frac{2GM}{r_E c^2}\right)\left(1 + \frac{1}{2}\frac{2GM}{r_R c^2}\right)$$

$$= 1 + \frac{GM}{c^2}\left\{\frac{1}{r_R} - \frac{1}{r_E}\right\}$$

which can be rewritten as

$$\frac{\nu_R}{\nu_E} - 1 = \frac{\Delta\nu}{\nu_E} \simeq \frac{GM}{c^2}\left\{\frac{1}{r_R} - \frac{1}{r_E}\right\}$$

So in this case $r_R > r_E$ means that $\Delta\nu$ is negative & thus the photons are red shifted.

This argument can actually be applied to any line element that can be written as

$$c^2 d\tau^2 = g_{00}(x^k)\, dt^2 + g_{ij}(x^k)\, dx^i dx^j$$

where the x^k dependence indicates that there is no time dependence in the metric components – the spacetime is static. We then proceed along the same line of reasoning given and arrive at

$$\frac{\nu_R}{\nu_E} = \left[\frac{g_{00}(x^k_E)}{g_{00}(x^k_R)}\right]^{1/2}$$

(Chap 12)

The Energy-Momentum tensor

Thus far we have only considered vacuum solutions for GR. The full field equations include the contribution non vacuum components, such as fluids or electric fields, to the curvature of spacetime. Obviously, we must represent these contributions using a tensor.

To begin the discussion, it is easiest to first think of non-interacting pressureless matter, which is known as dust. This is not a trivial solution, as cold dark matter in cosmology is essentially dust when viewed on cosmological scales. We can assume that each point of the dust field will move with a 4-velocity

$$u^a = \frac{dx^a}{d\tau}$$

The second component we need is a scalar field, $\rho_0(x)$, describing the density felt by an observer moving with the flow.

Since the Einstein tensor is rank 2, we need to construct from u^a & $\rho_0(x)$ a rank 2 tensor. The simplest rank 2 tensor we can form is

$$T^{ab} = \rho_0(x) u^a u^b$$

To see what this tensor represents it is easiest to use SR for a moment. In units of $c=1$, we have already seen

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \gamma(1, u^i) \quad \text{where } \gamma = (1-u^2)^{-1/2} \text{ and } u^i = \frac{dx^i}{dt}$$

note: we are following D'Inverno by writing the time derivatives of x^i as u^i; previously we used v^i

This also defines the relationship between proper time & coordinate time

$$d\tau^2 = \gamma^{-2} dt^2$$

We can now examine the general structure of T^{ab}. Consider T^{00}:

$$T^{00} = \rho_0(x) \frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = \rho_0\left(\frac{dt}{d\tau}\right)^2 = \rho_0 \gamma^2$$

To interpret this result, recall that the zeroth component of the momentum 4 vector, p^0 includes a factor of γ. We mentioned that this can be used to define the concept of relativistic mass $m\gamma$. The second γ comes from the length contraction associated with a moving observer. For a moving volume, length contraction will mean that the volume decreases by a factor of γ.