Solving The Relativistic version of Binet's Equation

Last lecture we arrived at

$$\frac{d^2u}{d\phi^2} + u = \frac{mc^2}{h^2} + 3mu^2$$

Comparing the ratio of terms of the RHS:

$$\frac{3mu^2}{mc^2/h^2} = \frac{3u^2h^2}{c^2}$$

Note: D'Inverno talks about $3h^2/r^2$ being small, although since this corresponds to the ratio of two terms on the RHS it should be dimensionless. Thus it should be clear there is a factor of $1/c^2$ missing, due to $c=1$ assumption. Nonetheless, the ratio is about 10^{-7}.

Thus the second term is small compared to the first. This suggests we consider using perturbation theory. If we let $\varepsilon = \frac{3m^2c^2}{h^2}$ (check this is dimensionless)

then
$$u'' + u = \frac{mc^2}{h^2} + \varepsilon\left(\frac{u^2h^2}{mc^2}\right) \qquad (1)$$

So now we look for first order solutions of the form

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2)$$

Substituting into (1)

$$u_0'' + \varepsilon u_1'' + u_0 + \varepsilon u_1 = \frac{mc^2}{h^2} + \varepsilon\left(\frac{u^2h^2}{mc^2}\right).$$

Expanding u^2 on RHS

$$u_0'' + \varepsilon u_1'' + u_0 + \varepsilon u_1 = \frac{mc^2}{h^2} + \frac{\varepsilon h^2}{mc^2}\left\{u_0^2 + \cancel{2\varepsilon u_1 u_0 + \varepsilon^2 u_1^2}\right\}$$

Dropping factor at $O(\varepsilon^2)$ and higher & rearranging

$$u_0'' + u_0 - \frac{mc^2}{h^2} + \varepsilon\left\{u_1'' + u_1 - \frac{u_0^2 h^2}{mc^2}\right\} = 0 \qquad \text{—— (2)}$$

Since ε is a problem dependent parameter that depends on m^2 & h^2 the zero'th order part & first order parts must be separately zero.
Therefore for the zero'th order part

$$u_0'' + u_0 = \frac{mc^2}{h^2}$$

Which is just ~~the~~ Binet's equation with solution

$$u_0 = \frac{mc^2}{h^2}(1 + e\cos\phi)$$

Remember that $m \equiv \frac{GM}{c^2}$ so $mc^2 \equiv GM \simeq \mu$

The first order equation is

$$u_1'' + u_1 = \frac{u_0^2 h^2}{mc^2}$$

So we substitute for u_0 to get

$$u_1'' + u_1 = \frac{m^2c^4}{h^4}\frac{h^2}{mc^2}(1 + e\cos\phi)^2 = \frac{mc^2}{h^2}(1+e\cos\phi)^2$$

$$= \frac{mc^2}{h^2}\left\{1 + 2e\cos\phi + e^2\cos^2\phi\right\}$$

$$= \frac{mc^2}{h^2}\left(1 + \tfrac{1}{2}e^2\right) + \frac{2mc^2}{h^2}e\cos\phi + \frac{mc^2e^2}{2h^2}\cos 2\phi$$

We then look for solutions

$$u_1 = A + B\phi \sin\phi + C\cos 2\phi$$

and we find (exercise)

$$A = \frac{mc^2}{h^2}\left(1 + \frac{1}{2}e^2\right) \qquad B = \frac{mc^2 e}{h^2} \qquad C = -\frac{mc^2 e^2}{6h^2}$$

Thus to first order in ε

$$u \simeq u_0 + \frac{\varepsilon mc^2}{h^2}\left\{1 + e\phi\sin\phi + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\phi\right)\right\}$$

Since the $e\phi\sin\phi$ terms grows linearly in ϕ we can neglect the other terms as being small (ie we choose some point when this is true) If we neglect the other terms & substitute for u_0

$$u = \frac{mc^2}{h^2}\left\{1 + e\cos\phi + \varepsilon e\phi\sin\phi\right\}$$

Using $\cos(\phi - \varepsilon\phi) = \cos\phi + \varepsilon\phi\sin\phi + \ldots$
we find

$$u = \frac{mc^2}{h^2}\left\{1 + e\cos(\phi(1-\varepsilon))\right\}$$

The length of the period is given by $2\pi = \phi(1-\varepsilon)$

$$\Rightarrow \quad \phi = \frac{2\pi}{1-\varepsilon} \simeq 2\pi(1+\varepsilon)$$

So the orbit must travel an extra $2\pi\varepsilon$ to return to same point (the perihelion).

We can roughly calculate $2\pi\varepsilon$ using results from Kepler's 3^{rd} Law:

$$2\pi\varepsilon = 2\pi \cdot \frac{3m^2c^2}{h^2} = \frac{6\pi (Gm)^2}{c^2h^2}$$

KL3: $Gm \simeq \frac{4\pi^2 a^3}{T^2}$ & $h^2 = \mu a(1-e^2) (= \mu l)$

Which gives

$$2\pi\varepsilon \simeq \frac{24\pi^3 a^2}{c^2T^2(1-e^2)} \quad \text{as } \mu \simeq GM.$$

1992: After accounting for precession due to other planets, the error between the actual precession & ~~observed~~ calculated value is 43.1 ± 0.5 arc seconds per century. GR predicts that the contribution should be 43.0

2007: Estimates are now in the 4^{th} digit of precision

Arc sec / century	Cause
531.4	Tugs of other planets
0.0254	Oblateness of Sun
42.98 ± 0.04	GR
574.4	Total calculated
574.1	Total observed

(After accounting for precession of the Equinoxes.)

Deflection of light (Gravitational lensing)

Light rays travel on null geodesics so we cannot use τ as our parameter. We can keep the same form of the Lagrangian but $\cdot$ will represent d/dw where w is our affine parameter. Hence the E-L equations will be

$$\frac{d}{dw}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$$

The procedure to find the geodesics is exactly the same as before, apart from one point. The metric geodesic will now be null ie

$$c^2\left(1-\frac{2m}{r}\right)\dot{t}^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0$$

whereas before it was c^2. We can use the same solutions for $\dot{t}$ & $\dot{\phi}$

$$\left(1-\frac{2m}{r}\right)\dot{t} = k$$

$$r^2\dot{\phi} = h$$

So again using $\dot{r}^2 = \left[\frac{d}{dw}\frac{1}{u}\right]^2 = \left[-\frac{1}{u^2}\frac{d\phi}{dw}\frac{du}{d\phi}\right]^2 = \left(-h\frac{du}{d\phi}\right)^2$

& substituting for $\dot{t}$ & $\dot{\phi}$ we get

$$\frac{c^2k^2}{h^2} - \left(\frac{du}{d\phi}\right)^2 - u^2 + 2mu^3 = 0$$

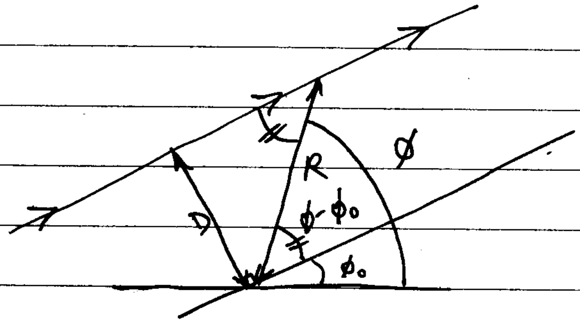
Differentiate wrt ϕ to get

$$\frac{d^2u}{d\phi^2} + u = 3mu^2$$

If we let $m \to 0$ (ie reduce to S.R.) then

$$\frac{d^2u}{d\phi^2} + u = 0$$

which has solution $u = \frac{1}{r} = \frac{1}{D}\sin(\phi - \phi_0) \Rightarrow r = D\,\mathrm{cosec}(\phi - \phi_0)$



If $r = \frac{D}{\sin(\phi - \phi_0)}$ point of closest approach is $r = D$

ie $\phi - \phi_0 = \frac{\pi}{2}$

ie a straight line in circular polars.

Let's look for another perturbative solution.

As $u \to$ large then the "object" (photon) is close by and we need m to be small. So we take $3m$ as the perturbation parameter (equivalently mu is small). Then

$$u = u_0 + \varepsilon u_1 \quad \text{and} \quad \frac{d^2u}{d\phi^2} + u = \varepsilon u^2$$

gives

$$u_0'' + u_0 + \varepsilon u_1'' + \varepsilon u_1 = \varepsilon u_0^2 + \text{h.o.t.}$$

and so

$$u_0'' + u_0 = 0 \qquad \text{(solved above)}$$

$$u_1'' + u_1 = u_0^2 = \frac{\sin^2\phi}{D^2} \qquad (\text{dropping } \phi_0)$$

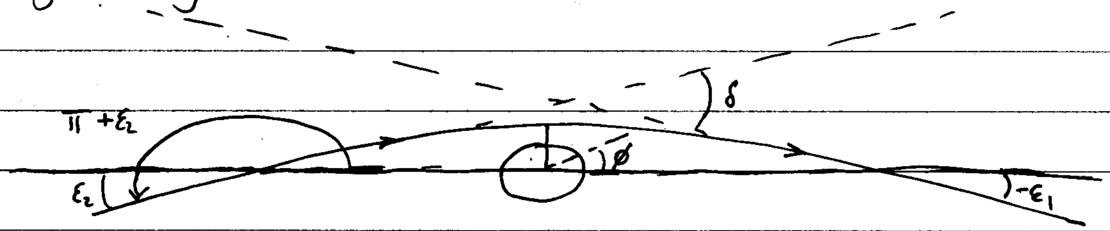
This has solution (check) $(1 + C\cos\phi + \cos^2\phi)/3D^2$

where C is a constant of integration.

$$\therefore\ u \simeq \frac{\sin\phi}{D} + \frac{m\left(1 + C\cos\phi + \cos^2\phi\right)}{D^2} \qquad \text{——— (A)}$$

So the second term represents a deviation from straight line motion.

To analyse further: as $r \to \infty$ $u \to 0$, so the RHS of (A) must go to zero as well. There will be two limiting situations for $r \to \infty$, the angle associated with light ray coming in & that associated with it going away:



The angle of the out going path is $-\varepsilon_1$, while that of the incoming path is $\pi + \varepsilon_2$.

We can substitute both these angles into (A) to find limits on ε_1 & ε_2; using the small angle approximation we find (remember $\cos(\pi + x) = -1 + x^2 + \ldots$)

$$-\frac{\varepsilon_1}{D} + \frac{m}{D^2}(2 + C) = 0 \qquad \text{——— (2)}$$

$$-\frac{\varepsilon_2}{D} + \frac{m}{D^2}\left(1 - C + 1\right) = -\frac{\varepsilon_2}{D} + \frac{m}{D^2}\left(2 - C\right) = 0 \qquad \text{——— (3)}$$

We can add (2) & (3) to get

$$-\frac{\varepsilon_1}{D} - \frac{\varepsilon_2}{D} + \frac{4m}{D^2} = 0$$

$$\Rightarrow \quad \delta = \varepsilon_1 + \varepsilon_2 = \frac{4m}{D} = \frac{4GM}{Dc^2}$$ (after substituting for $m = \frac{GM}{c^2}$)

Referring to the diagram it should be clear that placing a massive object in the centre of a star field will make the stars appear to move out.

In 1919 Sir Arthur Eddington measured deflections of stars during a solar eclipse & reported that the results confirmed Einstein's theory. This catapulted Einstein on a path to fame. In reality the measurements are extremely hard to do accurately & Eddington significantly underestimated his errors.

More on the Schwarzschild Geometry

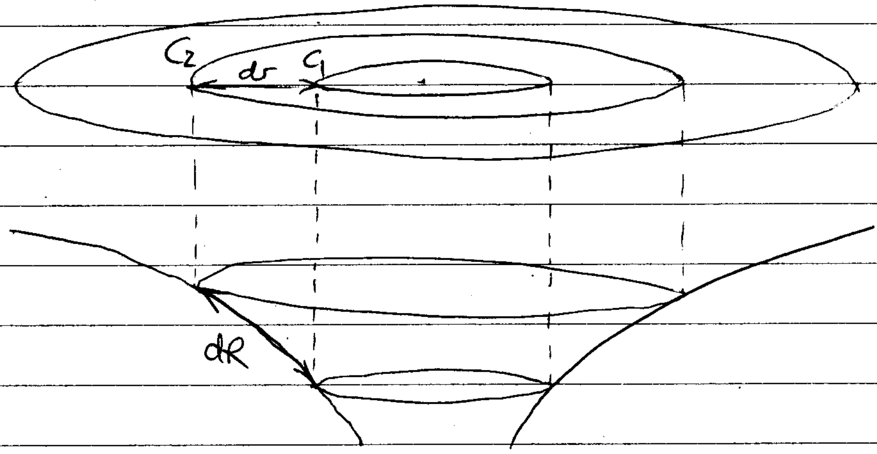
For the moment ignoring $r < 2m$, recall that the line element is given by

$$c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

so that, as stated earlier, we cannot relate r exactly to a radial distance. For lines of constant θ, ϕ we can instead write

$$dR = \left(1 - \frac{2m}{r}\right)^{-1/2} dr$$

where R is the physical distance coordinate (radially). We can relate the change in dr to that in dR with the following diagram:



Each disk represent a sphere & the physical separation dR is larger than dr. This is directly indicated by the lower curved surface. However, the circumference of each circle is the same in both coordinate systems – only the radial coordinate is changed. This should be clear from the fact that

There is no factor of $(1-\frac{2m}{r})$ in the angular part of the metric $r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$.

What about time? At a fixed spatial point we have

$$d\tau = \left(1-\frac{2m}{r}\right)^{1/2} dt$$

As $m \longrightarrow 0$ so $d\tau = dt$, but for non-zero m we must have $d\tau < dt$. Obviously, since $d\tau$ depends on r, an ensemble of clocks at different points in Schwarzschild geometry do not all agree on the passage of time.

It is tempting to draw similarities between the change in length & time in Schwarzschild geometry & that produced by Lorentz transforms:

S.R. v = constant	Schwarzschild, r variable
$dl = dl_0\left(1-\frac{v^2}{c^2}\right)^{1/2}$	$dr = dR\left(1-\frac{2m}{r}\right)^{1/2}$
$dt = d\tau\left(1-\frac{v^2}{c^2}\right)^{-1/2}$	$dt = d\tau\left(1-\frac{2m}{r}\right)^{-1/2}$

But remember - Schwarzschild geometry has an r dependence & a different coordinate system for Schwarzschild geometry would have different relationships for dR & $d\tau$.

Examples: Suppose a stick of length 1m lies radially in the field of an object of mass m, where $m/r = 10^{-2}$. What "coordinate length" does it have?

Assuming ΔR is small, and $\Delta R = 1\text{m}$ then

$$\Delta r = \Delta R\left(1-\frac{2m}{r}\right)^{1/2}$$

$$\Rightarrow \quad \Delta r = 0.99\text{m}$$

Similarly, suppose the radial coordinate ends of a stick are r_2 & r_1. What is the physical length of the stick in Schwarzschild geometry?

$$dR = dr\left(1-\frac{2m}{r}\right)^{-1/2}$$

Thus

$$\int_{R_1}^{R_2} dR = \int_{r_1}^{r_2} dr\left(1-\frac{2m}{r}\right)^{-1/2} = \int_{r_1}^{r_2} dr\left(\frac{r-2m}{r}\right)^{-1/2}$$

$$= \left[r^{1/2}(r-2m)^{1/2} + 2m\ln\left\{r^{1/2}+(r-2m)^{1/2}\right\}\right]_{r_1}^{r_2}$$

In the limit $m \to 0$ the $R_2 \to r_2$ & $R_1 \to r_1$.

Newtonian Limit of GR (Section 12.9)

At this point it is useful to consider the Newtonian limit of GR. The approach to the derivation is perturbative, and we shall perturb the metric. Let's assume there exists a privileged coordinate system

$$x^\alpha \equiv (x^0, x^1, x^2, x^3) = (x^0, x^i) = (ct, x, y, z)$$

such that the metric is

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \varepsilon h_{\alpha\beta} \qquad \text{———} (A)$$

Hence the geometry is close to Minkowski space, but there is a perturbation $\varepsilon h_{\alpha\beta}$ where ε is order v/c where v is the typical velocity of an object. Further these velocities are such that $v \ll c$.

Thus if a body moves δx^i in space then

$$\delta x^i \approx v\, dt = \left(\frac{v}{c}\right) c\, \delta t \simeq \varepsilon\, \delta x^0$$

So that $\dfrac{\varepsilon}{\delta x^i} \sim \dfrac{\varepsilon}{\varepsilon \delta x^0} = \dfrac{1}{\delta x^0}$

ie derivatives wrt dx^0 are much smaller than spatial derivatives,

$$\varepsilon \frac{\partial f}{\partial x^i} \sim \frac{\partial f}{\partial x^0} \qquad \text{——— (B)}$$

Which is called the <u>slow motion approximation</u>.

~~Under this assumption~~

The next step is to conside a particle moving on a timelike geodesic under the above considerations. So we must examine

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0$$

To relate to Newtonian Theory we want to replace proper time τ with coordinate time t. So we use the line element

$$ds^2 = c^2 d\tau^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

Let's assume $h_{\alpha\beta}$ diagonalizes as well (not necessary but makes algebra easier). Then

$$c^2 d\tau^2 = (\eta_{\alpha\beta} + \varepsilon h_{\alpha\beta})\, dx^\alpha dx^\beta$$

$$= (\eta_{00} + \varepsilon h_{00})\, dx^{0\,2} + (\eta_{11} + \varepsilon h_{11})\, dx^{1\,2} + (\eta_{22} + \varepsilon h_{22})\, dx^{2\,2} + (\eta_{33} + \varepsilon h_{33})\, dx^{3\,2}$$

$$= c^2 dt^2 - dx^{1\,2} - dx^{2\,2} - dx^{3\,2} + \varepsilon \{ h_{00}\, dx^{0\,2} + h_{11}\, dx^{1\,2} + h_{22}\, dx^{2\,2} + h_{33}\, dx^{3\,2} \}$$

For particle moving with velocity $\vec{v}$

$$dx^1 = v_1\, dt \quad dx^2 = v_2\, dt \quad dx^3 = v_3\, dt$$

(lower case since we use v^2)

Thus

$$c^2 d\tau^2 = \left\{ c^2 - v^2 + \varepsilon \left\{ c^2 h_{00} + h_{11} v_1^2 + h_{22} v_2^2 + h_{33} v_3^2 \right\} \right\} dt^2$$

$$\Rightarrow \quad \frac{dt}{d\tau} = \frac{1}{\left\{ 1 - \varepsilon^2 + \varepsilon \left(h_{00} + h_{11} \frac{v_1^2}{c^2} + h_{22} \frac{v_2^2}{c^2} + h_{33} \frac{v_3^2}{c^2} \right) \right\}^{1/2}}$$

$$\Rightarrow \quad \frac{dt}{d\tau} = 1 + O(\varepsilon) \qquad \frac{d^2 t}{d\tau^2} \simeq 0.$$

For the $\frac{d^2 x^\alpha}{d\tau^2}$ term we get

$$\frac{d^2 x^\alpha}{d\tau^2} = \left(\frac{dt}{d\tau} \right)^2 \frac{d^2 x^\alpha}{dt}$$

and if we consider only the spatial coordinate geodesic equations

$$\frac{d^2 x^i}{d\tau^2} = (1 + O(\varepsilon)) \frac{d^2 x^i}{dt^2}$$

For the connection coefficients, we break up the sum after replacing $d/d\tau$ with d/dt:

$$\Gamma^i{}_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = \Gamma^i{}_{\beta\gamma} \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} (1 + O(\varepsilon))$$

$$= \left(\Gamma^i{}_{00} c^2 + 2 \Gamma^i{}_{0j} c \frac{dx^j}{dt} + \Gamma^i{}_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} \right) \times (1 + O(\varepsilon))$$

The next step is to write down the connection coefficients in terms of $h_{\alpha\beta}$ via the metric connection.

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\gamma g_{\beta\delta} + \partial_\beta g_{\gamma\delta} - \partial_\delta g_{\beta\gamma} \right)$$

Thus

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}(\eta^{\alpha\delta} + \varepsilon h^{\alpha\delta})(\partial_\gamma \varepsilon h_{\beta\delta} + \partial_\beta \varepsilon h_{\gamma\delta} - \partial_\delta \varepsilon h_{\beta\gamma})$$

$$= \frac{1}{2}\eta^{\alpha\delta}\varepsilon\,(\partial_\gamma h_{\beta\delta} + \partial_\beta h_{\gamma\delta} - \partial_\delta h_{\beta\gamma}) + O(\varepsilon^2)$$

So the connection coefficient are order ε and higher. Using the above formula we find

$$\Gamma^{i}{}_{00} = -\frac{1}{2}\varepsilon\left(2\frac{\partial h_{0i}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^i}\right)$$

Under the slow motion approximation $\frac{\partial h_{0i}}{\partial x^0} \propto \varepsilon \frac{\partial h_{0i}}{\partial x^i}$

$$\therefore \Gamma^{i}{}_{00} = +\frac{1}{2}\varepsilon\frac{\partial h_{00}}{\partial x^i} + O(\varepsilon^2)$$

For the $\Gamma^{i}{}_{0j}\, c\frac{dx^j}{dt}$ term, it must have order

$$O(\varepsilon) \times c\,v_j \simeq O(\varepsilon)\,c^2 O(\varepsilon) = O(\varepsilon^2)$$

while $\Gamma^{i}{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$ is

$$O(\varepsilon)\, v_j v_k \simeq O(\varepsilon)\,c^2 O(\varepsilon) O(\varepsilon) = O(\varepsilon^3)$$

Thus the full spatial geodesic equation are

$$(1+O(\varepsilon))\frac{d^2x^i}{dt^2} + \frac{c^2}{2}\varepsilon\frac{\partial h_{00}}{\partial x^i}(1+O(\varepsilon)) + O(\varepsilon^2) + O(\varepsilon^3) = 0$$

We can replace $\varepsilon\frac{\partial h_{00}}{\partial x^i}$ with $\frac{\partial}{\partial x^i}\{\eta_{00} + \varepsilon h_{00}\} = \frac{\partial}{\partial x^i} g_{00}$

$$\therefore (1+O(\varepsilon))\frac{d^2x^i}{dt^2} + \frac{c^2}{2}\frac{\partial g_{00}}{\partial x^i}(1+O(\varepsilon)) + O(\varepsilon^2) + O(\varepsilon^3) = 0$$

If we relate to the Newtonian acceleration in a gravitational field

$$a^i = -\frac{\partial \phi}{\partial x^i}$$

[But bear in mind that g_{00} must obey the following: as masses go to zero $g_{00} \to 1$, similarly as distances from sources go to infinity $g_{00} \to 1$. While $\phi \to 0$.]

To give

$$\frac{\partial}{\partial x^i}\left(\frac{c^2}{2} g_{00}\right) = \frac{\partial \phi}{\partial x^i}$$

$$\therefore \quad g_{00} = 1 + \frac{2\phi}{c^2} + O(\varepsilon).$$

Which is called the weak field limit. This is an extremely useful result that virtually underpins the usage of a pseudo-Newtonian approach in studies of gravitational structures on large-scales.

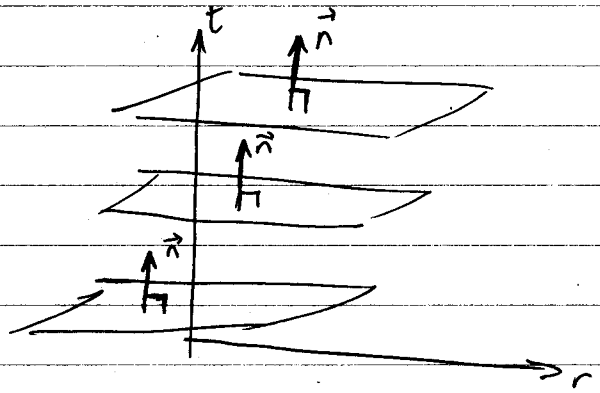
(Chap 16).

Non-rotating Black Holes

Characterization of Coordinates

General Relativity will present us with situations where time & spatial coordinates appear to have swapped roles. It is therefore useful to characterize the precise nature of a hypersurface $x^{(\alpha)} = \text{constant}$. (In this case we are using ( ) to represent the index is to be held constant).

For example we can consider hypersurfaces of constant t



Each surface will have a normal vector n, where

$$n_\beta = \delta^{(\alpha)}_\beta$$

It should be clearly that a surface of $t = \text{constant}$ has a normal $(1,0,0,0)$.

The contravariant version of the normal vector is then

$$n^\gamma = g^{\gamma\beta} n_\beta = g^{\gamma\beta} \delta^{(\alpha)}_\beta = g^{\gamma(\alpha)}$$

We can then categorize the nature of the surface by considering whether the normal is timelike, spacelike or null. Recall from definitions in S.R. if $n^2 = n^\gamma n_\gamma$ then

$$n^2 > 0 \Rightarrow \text{timelike vector}$$
$$n^2 = 0 \Rightarrow \text{null vector}$$
$$n^2 < 0 \Rightarrow \text{spacelike vector}$$

If we write out $n^2 = n^\gamma n_\gamma$ then

$$n^2 = n^\gamma n_\gamma = \delta^{(\alpha)}_\gamma g^{\gamma(\alpha)} = g^{(\alpha)(\alpha)} \qquad \text{(no summation)}$$

So the nature of the coordinate hypersurfaces is described as follows:

$$g^{(\alpha)(\alpha)} > 0 \Rightarrow \text{timelike coord. hypersurface}$$
$$g^{(\alpha)(\alpha)} = 0 \Rightarrow \text{null " "}$$
$$g^{(\alpha)(\alpha)} < 0 \Rightarrow \text{spacelike " "}$$

For the -2 signature we use we might expect 1 timelike surface & 3 spacelike. However, GR allows this signature to change and it is common to meet situations with one null & three spacelike coord. hypersurfaces, and two null & two spacelike coord. hypersurfaces.

For the Schwarzschild metric we have an immediate & interesting consequence. Since $g_{00} = (1 - \frac{2m}{r})$

$$\Rightarrow \quad g^{00} = (1 - \tfrac{2m}{r})^{-1}$$

Which is >0 for $r > 2m$, (and hence timelike)

although for $r < 2m$ $g^{00} < 0$ ∴ it becomes a spacelike coordinate inside $r = 2m$.
The same analysis applies to the g^{11} component outside $r = 2m$, r is spacelike but inside $r = 2m$ it becomes timelike. Therefore interior to $r = 2m$ the radial & time coordinates swap their roles.

The Nature of Singularities

For the Schwarzschild metric $\theta = 0, \pi$ marks a degeneracy since $\sin\theta = 0$ at both these points. We could remove this degeneracy though by introducing x, y, z coords

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

Such points are called coordinate singularities because they represent a defficiency of the coordinates rather than being physical.
Similarly for $r = 2m$ the Schwarzschild metric has divergent time or space components. This is actually a removable coordinate singularity (we'll see how this is done later).
Although we won't discuss it formally the scalar defined by

$$R_{abcd} R^{abcd} = 48 m^2 r^{-6}$$

tells us that the Riemann tensor is not infinite at $r = 2m$. However, as $r \to 0$ this invariant does indeed diverge.
Thus $r \to 0$ is a true (or physical, or curvature) singularity.

Light Rays in the Schwarzschild Metric

Lets examine the properties of radial geodesics $\dot\theta = \dot\phi = ds^2 = 0$. We have already completed the derivation of the geodesic in the section on the deflection of light (although $\dot\phi$ was free).
In our new situation $\dot\phi$ is removed from the metric geodesic to give

$$\left(1-\frac{2m}{r}\right)\dot t^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot r^2 = 0$$

While the E-L equation for $\dot t$ is as before,

$$\left(1-\frac{2m}{r}\right)\dot t = k$$

Substituting into the metric geodesic gives

$$\left(1-\frac{2m}{r}\right)^{-1}k^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot r^2 = 0$$

$$\Rightarrow \quad \dot r^2 = k^2$$

$$\therefore \quad \dot r = \pm k$$

($\Rightarrow r = kw + a$ where w is the affine parameter)

Let's derive a direct solution for t as a function of r

$$\frac{dt}{dr} = \frac{dt/dw}{dr/dw} = \frac{k}{\left(1-\frac{2m}{r}\right)k} = \frac{r}{r-2m}$$

and integration by parts gives

$$t = r + 2m \ln|r-2m| + \text{constant} \qquad (45)$$

The mod is important.

For a region where $r > 2m$ the it should be clear that $\frac{dr}{dt} > 0$ and $\frac{dr}{dt} \to 1$ as $r \to \infty$.

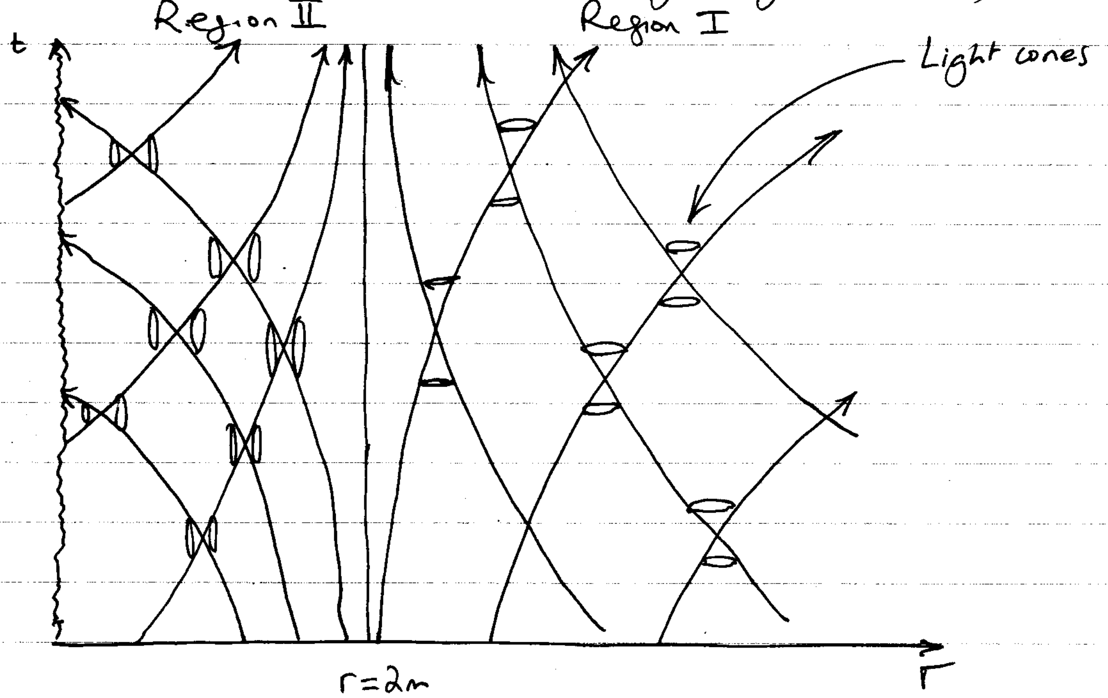
In this case r increases as t increases, so this is an outgoing geodesic. Equation (A) actually defines a congruence of outgoing geodesics.

If we take the negative solution for $\dot{r}$, then

$$t = -(r + 2m \ln |r - 2m| + \text{constant})$$

which are ingoing geodesics.

We can then the structure of the congruence of geodesics (remember this is for fixed θ, ϕ)



As $r \to \infty$ the geodesics have a $45°$ slope relative to the r axis. Also, inside $r = 2m$ the light cones "tip over" because t & r swap roles. One problem with this diagram it seems to suggest someone falling into the Schwarzschild radius takes an infinite amount of time to get there!
(Which is wrong!)

Paths of Radially Infalling Particles

For (massive) radially infalling particles we can take proper time as the affine parameter. The geodesic equations are then the same as for light rays except that the metric geodesic is no longer null. Thus in units of $c = 1$ we find

$$\left(1 - \frac{2m}{r}\right)\dot{t} = k$$

$$\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 = 1$$

where dot denotes $d/d\tau$. By setting the constant of integration to $k = 1$ we get

$$\dot{t} = \left(1 - \frac{2m}{r}\right)^{-1}$$

and for $\dot{r}$ after substituting for $\dot{t}$

$$\left(1 - \frac{2m}{r}\right)^{-1} - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 = 1$$

$$\Rightarrow \qquad \dot{r}^2 = \frac{2m}{r} \qquad \Rightarrow \dot{r} = 0 \text{ at } r = \infty$$

Taking the negative square root for the incoming particle

$$\frac{d\tau}{dr} = -\left(\frac{r}{2m}\right)^{1/2}$$

$$\Rightarrow \quad \int_{\tau_0}^{\tau} d\tau = -\frac{1}{(2m)^{1/2}}\int_{r_0}^{r} r^{1/2}\,dr$$

$$\Rightarrow \quad \tau - \tau_0 = \frac{2}{3(2m)^{1/2}}\left(r_0^{3/2} - r_0\right) \qquad \text{(A)}$$

Which is actually the same result as for Newtonian theory (replacing τ by t you can then easily see this solution solves $\frac{d^2r}{d\tau^2} = -\frac{Gm}{r^2}$).
Thus there isn't a "problem" with an object falling through the Schwarzschild radius (the body can reach $r=0$ in finite proper time).

However, in terms of coordinate time t,

$$\frac{dt}{d\tau} = -\left(\frac{r}{2m}\right)^{1/2}\left(1-\frac{2m}{r}\right)^{-1}$$

which has solution

$$t - t_0 = \frac{-2}{3(2m)^{1/2}}\left(r^{3/2} - r_0^{3/2} + 6mr^{1/2} - 6mr_0^{1/2}\right) + 2m \ln \frac{\left(r^{1/2}+(2m)^{1/2}\right)\left(r_0^{1/2}-(2m)^{1/2}\right)}{\left(r_0^{1/2}+(2m)^{1/2}\right)\left(r^{1/2}-(2m)^{1/2}\right)}$$

While for $r, r_0 \gg 2m$ this is similar to (A), around $r \simeq 2m$ it can be shown that

$$r - 2m = (r_0 - 2m)\, e^{-(t-t_0)/2m}$$

So as $t \to \infty$ r never crosses $2m$.

The resolution of this issue lies in what the coordinate time represents. Remember, t corresponds to the time coordinate of an observer far from the origin (away from fields etc). Therefore such an observer never sees the test body reach $r = 2m$. However, we do know that for an observer falling into the black hole, that they reach the singularity in a finite proper time – we must abandon coordinate time!

By using a new time coordinate we will also be able to extrapolate through the $r=2m$ coordinate singularity.

Eddington - Finkelstein Coordinates

Thus far the light rays we've looked at were not straight lines in r,t coordinates. We can make a coordinate transformation to straighten them. Since the equation of the outgoing rays is

$$t = r + 2m \ln |r-2m| + \text{constant}$$

If we let

$$t \longrightarrow \bar{t}(t,r) = t + 2m \ln (r-2m)$$

Then for an incoming ray of equation

$$t = -(r + 2m \ln |r-2m| + \text{const})$$

we get

$$\bar{t} = -r + \text{const}.$$

Differentiating the definition of $\bar{t}(t,r)$ we have

$$d\bar{t} = dt + \frac{2m}{(r-2m)} dr$$

& substituting for dt in the Schwarzschild line element gives

$$ds^2 = \left(1-\frac{2m}{r}\right) d\bar{t}^2 - \frac{4m}{r} d\bar{t}\, dr - \left(1+\frac{2m}{r}\right) dr^2 - r^2(d\theta^2 + \sin^2 d\phi^2)$$

Thus we have removed the coordinate singularity at $r=2m$. One might argue that using $\ln(r-2m)$ inside $r=2m$ is not allowed. However, the key point here is that the new metric is a valid description of Schwarzschild geometry inside $r=2m$. The map between the two coordinate systems is only applicable outside $r = 2m$. This idea of extending a coordinate system that works in one region to another is analogous to analytic extensions in complex analysis.

Note that this new coordinate system is not time symmetric (why?) !

We can simplify the metric further by introducing a "null coordinate"

$$v = t + r + 2m \ln(r - 2m)$$
$$= \bar{t} + r \qquad \text{———} (B)$$

with this new parameter we find the line element is

$$ds^2 = \left(1 - \frac{2m}{r}\right) dv^2 - 2\,dv\,dr - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

What do light rays look like in this coordinate system?
By setting $ds^2 = 0 = d\theta^2 = d\phi^2$ the metric geodesic is

$$0 = \left(1 - \frac{2m}{r}\right)\left(\frac{dv}{dw}\right)^2 - 2\,\frac{dv}{dw}\frac{dr}{dw}$$

where w is an affine parameter. Since $\frac{dv}{dw} = \frac{dr}{dw}\frac{dv}{dr}$

$$0 = \left(1 - \frac{2m}{r}\right)\left(\frac{dv}{dr}\right)^2 - 2\,\frac{dv}{dr}$$

which has solution

$$\frac{dv}{dr} = \begin{cases} 0 \\ \dfrac{2}{\left(1 - \frac{2m}{r}\right)} \end{cases}$$

What do these solutions correspond to?
Firstly, from the definition of v given in (B)

$$\frac{dv}{dr} = \frac{dt}{dr} + 1 + \frac{2m}{(r-2m)} = \frac{dt}{dr} + \frac{1}{(1-\frac{2m}{r})}$$

Then for the first of the two solutions, $\frac{dv}{dr} = 0$

$$0 = \frac{dt}{dr} + \frac{1}{(1-\frac{2m}{r})} \Rightarrow \frac{dt}{dr} = -\frac{1}{(1-\frac{2m}{r})}$$

Which is the equation of an <u>ingoing</u> null geodesic (at least for $r > 2m$ in r, t coordinates).

For the second solution, $\frac{dv}{dr} = \frac{2}{(1-\frac{2m}{r})}$ we get

$$\frac{dt}{dr} = \frac{1}{(1-\frac{2m}{r})}$$

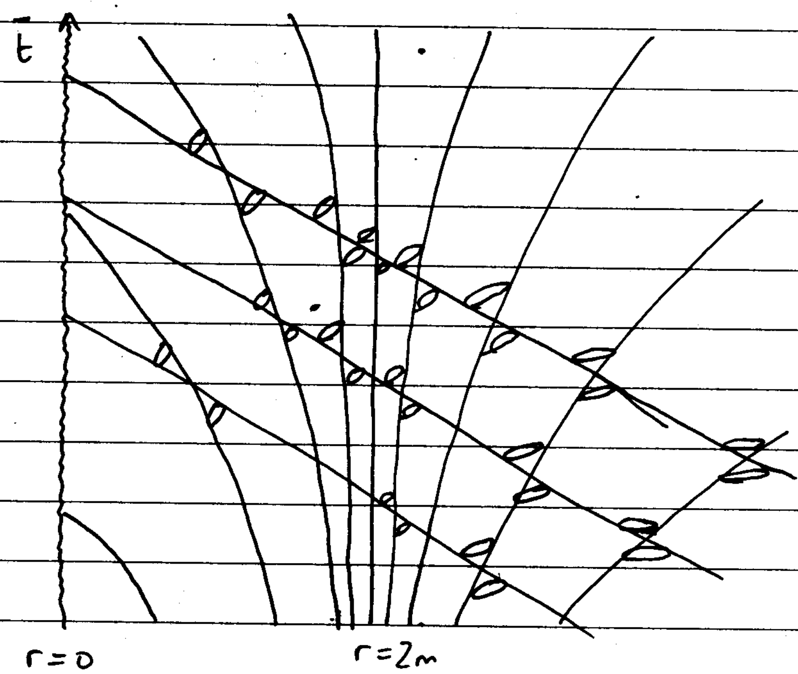
So outside $r = 2m$ $\frac{dt}{dr}$ is +ve and we take this to imply outgoing geodesic solutions.

Since $dv/dr = 0$ has solution v = constant, the ingoing geodesics are straight lines on a v, r graph.

While integration of $\frac{dv}{dr} = \frac{2}{(1-\frac{2m}{r})}$ gives

$$v = 2r + 4m \ln |r - 2m| + \text{constant}$$

Rather than drawing a spacetime diagram in v, r we'll instead go back to using $\bar{t}$. In this case the following diagram represents the spacetime structure of Schwarzschild geometry using E-F coordinates.



The closer we get to the singularity the more the light cones "tip over." At the $r=2m$ surface the outer part of the light cone (ie outward going photons) stay exactly on the $r=2m$ line.

Reverting back to 3 dimensions of space, it should be clear that $r=2m$ acts as a one-way "membrane." Material & light from $r>2m$ can pass through it, but no signal starting from within $r=2m$ can pass outside $r=2m$.
Thus the $r=2m$ sphere represent the boundary of events that can be observed at $r>2m$.
It is the event horizon.