The Correspondence Principle

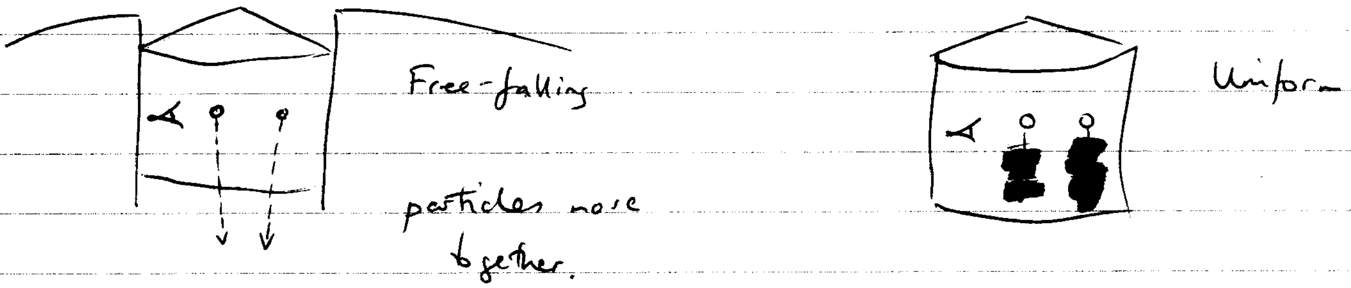
This principle might seem like "common sense" to some. Nonetheless, it explicitly details the realms of applicability of a given theory. For example quantum theory must reduce to classical physics when the quantum numbers describing the system are large. For GR we can summarize a number of limits with the following diagram

$$\begin{tikzcd}
\boxed{\text{General Relativity}} \arrow[r, "\text{weak field limit}"'] \arrow[d, "g \to 0"] & \boxed{\text{Newtonian Gravity}} \arrow[d, "g \to 0"] \\
\boxed{\text{Special Relativity}} \arrow[r, "\text{low speeds}"'] & \boxed{\text{Newtonian mechanics without gravity}}
\end{tikzcd}$$

The Vacuum Field Equations

We mentioned briefly in the discussion of the Equivalence Principle that over a long period of time the lift falling in the gravitational field would have subtle differences to the ~~accelerated~~ uniform motion lift. These differences are only noticeable if you move away from a given point - They are non-local.

Comparing the free-falling lift with the uniform motion lift:



The observer in the ff lift sees particles come together if given a sufficient amount of time. What we are actually observing is the change in the field rather than the field itself.

The idea that particles on different paths do not follow the same geodesic is called "geodesic deviation". It is not immediately obvious how this relates to field equations but we'll investigate that in this lecture.

Note I will only sketch the full GR approach as it is quite lengthy & involves a number of different concepts. We'll present the Newtonian version & the sketch the GR version. D'Inverno sections 10.3 & 10.4 gives details if you are interested.

For the Newtonian description of these two particles falling in gravity we use x^α to represent 3 vectors. This will allow us to relate to the tensor notation we have developed.

In $\mathbb{R}^3$, the line element is

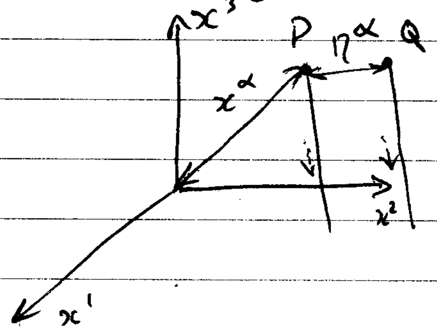
$$d\sigma^2 = dx^2 + dy^2 + dz^2$$
$$= g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

So the metric tensor in this case must be equivalent to the Kronecker delta:

$$g_{\alpha\beta} = \delta_{\alpha\beta}$$

Since we raise & lower indices using $g_{\alpha\beta}$ this means that there is no difference between the components of covariant & contravariant vectors in this geometry.

We now consider the paths of the two particles. They are falling in a vacuum with potential ϕ



The distance between the particles is described by $\eta^\alpha(t)$ the position of the first particle at P is $x_P^\alpha(t)$.

The position of the second particle is

$$x_Q^\alpha(t) = x_P^\alpha(t) + \eta^\alpha(t)$$

& η^α is called the connecting vector.

Taking units such that particle masses are 1, then

$$\vec{F} = m^I \vec{a} \quad \Rightarrow \quad F^\alpha = m^I \ddot{x}^\alpha$$

and

$$\vec{F} = -m^P \nabla\phi \quad \Rightarrow \quad F^\alpha = -m^P \partial^\alpha \phi$$

Note: $\partial^\alpha \phi = \delta^{\alpha\beta} \partial_\beta \phi$ and $\partial_\beta \phi$ is a component of $\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

Hence the equation of motion for the first particle is

$$1\,\ddot{x}^\alpha = -1\,(\partial^\alpha \phi)_P \quad \text{— (A)}$$

where the 1 & -1 are included for clarity. We drop them from now on.

The same analysis for the second particle gives

$$\ddot{x}^\alpha + \ddot{\eta}^\alpha = -(\partial^\alpha \phi)_Q \quad \text{— (B)}$$

The next step is to derive an equation for $\ddot{\eta}^\alpha$. This can be done by subtracting (A) from (B)

$$\ddot{\eta}^\alpha = -(\partial^\alpha \phi)_Q + (\partial^\alpha \phi)_P.$$

Since η^α is the connecting vector, we can expand $(\partial^\alpha \phi)_a$ using a Taylor series in η^α

$$(\partial^\alpha \phi)_a = (\partial^\alpha \phi(x^\alpha + \eta^\alpha))$$

$$= (\partial^\alpha \phi(x^\alpha))_p + (\eta^\beta \partial_\beta \partial^\alpha \phi(x^\alpha))_p$$

Hence

$$\ddot{\eta}^\alpha = -\eta^\beta \partial_\beta \partial^\alpha \phi(x^\alpha) \qquad \text{—— (c)}$$

By definition we set

$$K^\alpha{}_\beta = K_\beta{}^\alpha = \partial^\alpha \partial_\beta \phi$$

Then the evolution of the connecting vector is described by

$$\ddot{\eta}^\alpha + K^\alpha{}_\beta \eta^\beta = 0$$

This is the <u>Newtonian equation of deviation</u>

To see how this relates to Newton's field equation for gravity in a vacuum (the Laplace equation $\nabla^2\phi = 0$) we expand ∇^2 as follows

$$\nabla^2 = \partial^\alpha \partial_\alpha$$

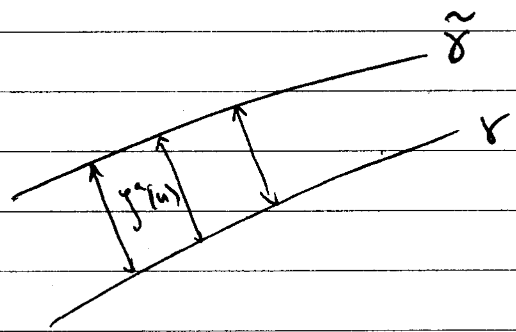
thus

$$\nabla^2 \phi = \partial^\alpha \partial_\alpha \phi = K^\alpha{}_\alpha$$

Therefore $K^\alpha{}_\alpha = 0$ (the "trace" of $K^\alpha{}_\beta$) is equivalent to $\nabla^2\phi = 0$. In this case $K^\alpha{}_\beta$ is said to be "trace free".

So the equation of deviation has embedded within it the field equation of gravity in a vacuum. If we would describe the same process in GR then we would find the field equation for GR.

In practice the GR derivation is quite lengthy & involves a number of concepts that would take time to explain. Hence we give a sketch.



Consider two geodesics γ & $\tilde{\gamma}$. Then for each geodesic we have the geodesic equation

$$\frac{d^2 x^a}{du^2} + \Gamma^a{}_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0$$

and a similar relationship for coords $\tilde{x}$ holds. If we expand $\tilde{\Gamma}^a{}_{bc}$ as a function of $\xi(u)$ (recall potential expansion!)

$$\tilde{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + \Gamma^a{}_{bc,d}\,\xi^d$$

Subtraction of the two geodesic equations with $\hat{\Gamma}^a{}_{bc}$ expanded gives

$$\frac{d^2\xi^a}{du^2} + \Gamma^a{}_{bc,d}\frac{dx^b}{du}\frac{dx^c}{du}\xi^d + \Gamma^a{}_{bc}\frac{dx^b}{du}\frac{d\xi^c}{du} + \Gamma^a{}_{bc}\frac{d\xi^b}{du}\frac{dx^c}{du} = 0$$

After a great deal of rearrangement & substitution

$$\frac{D^2\xi^a}{Du^2} + \left(\Gamma^a{}_{cd,b} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{be}\Gamma^e{}_{dc} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}\right)\xi^b\frac{dx^c}{du}\frac{dx^d}{du} = 0$$

or

$$\frac{D^2\xi^a}{Du^2} + R^a{}_{cbd}\,\xi^b\frac{dx^c}{du}\frac{dx^d}{du} = 0$$

where $R^a{}_{cbd}$ is the Riemann tensor.

Note: $R^a{}_{cbd} = -R^a{}_{cdb}$,

which gives 10.21 in D'Inverno.

This is still a 4-d equation so the spatial components must be "projected out" using a projection operator $h^a{}_b$ on ξ^a, these projected components are called $\eta^a = h^a{}_b\,\xi^b$.

The resulting equation becomes

$$\frac{D^2\eta^\alpha}{Du^2} + K^\alpha{}_\beta\,\eta^\beta = 0$$

and

$$K^\alpha{}_\beta = -R^a{}_{bcd}\,e^\alpha{}_a\frac{dx^b}{du}\frac{dx^c}{du}e_\beta{}^d$$

The $e^{\alpha}{}_{a}$ correspond matrices that allow us to extract components of η^{α} in the given frame.

D'Inverno shows in section 10.5 that the vanishing trace

$$R^{a}{}_{bcd}\, e^{\alpha}{}_{a} \frac{dx^{b}}{du} \frac{dx^{c}}{du} e^{d}{}_{\alpha} = 0$$

(note you can replace u with τ) leads to the requirement

$$R_{ab} = 0$$ where R_{ab} is the Ricci tensor.

This also implies $G_{ab} = 0$ in a vacuum.

Since $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$

$$\Rightarrow \quad R_{ab} = \frac{1}{2} R g_{ab}$$

$$\Rightarrow \quad R^{a}{}_{b} = \frac{1}{2} R g^{a}{}_{b} = \frac{1}{2} R \delta^{a}{}_{b}$$

Contracting with δ^{b}_{a} gives

$$R = R^{b}{}_{b} = \frac{1}{2} R g^{b}{}_{b} = 2R$$

Which is only true iff $R = 0$

$$\therefore \quad G_{ab} = R_{ab} = 0.$$

Hence $R_{ab} = 0$ is equivalent to $G_{ab} = 0$

This defines the vacuum field equations for General Relativity:

$$G_{ab} = 0.$$

Don't be fooled by the simplicity of the notation.

Due to symmetry, this is 10 equations in the second derivative of the metric tensor, and further the equations are coupled!

The full equations also include a source term $T^{\mu\nu}$ which is the energy-momentum tensor.

$$G_{ab} = k T_{ab}$$

where k is a coupling term.

We'll look at the full equations in more detail later. First we look at the Schwarzschild solution of the vacuum field equations.

Solving the Vacuum FE's

The equation $G_{\mu\nu} = 0$ at first appearances seems very difficult to solve. However, if we look for special solutions that exhibit symmetries, such as isotropy, we can simplify finding the solution. The first solution to the EFE's calculated by K. Schwarzschild in 1916 takes this approach.

The simplest solution we can look for is one corresponding to a stationary, (strictly speaking static) point mass. Isotropy necessarily means this is a spherically symmetric solution. We may also desire the field to be asymptotically flat, ie far away from the point mass spacetime looks like SR - Minkowski space.

Asymptotic flatness is represented by a metric

$$ds^2 = c^2 dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

which we can rewrite using spherical polar coordinates

$$x^1 = r \sin\theta \cos\phi$$
$$x^2 = r \sin\theta \sin\phi$$
$$x^3 = r \cos\theta$$

to give

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \quad \text{——(A)}$$

Requiring that we look for a <u>static</u> solution means that we do not expect any explicit dependence on time in the metric components. Secondly, we would also expect there not to be any $dx^0 dx^i$ $(i \neq 0)$ cross terms. To see this, consider two points (x^0, x^1, x^2, x^3) & $(x^0 + dx^0, x^1 + dx^1, x^2, x^3)$. Then for a generic line element:

$$ds^2 = g_{00}(dx^0)^2 + 2g_{01}\, dx^0 dx^1 + g_{11}(dx^1)^2$$

Under time reversal $x^0 \to x'^0 = -x^0$ the g_{ab} are unchanged, but

$$ds^2 = g_{00}(dx^0)^2 - 2g_{01}\, dx^0 dx^1 + g_{11}(dx^1)^2$$

This is clearly not consistent with our assumption of a static solution since we would expect ds^2 to be unchanged. To make this true we must not have any cross terms, ie, g_{01} must vanish. The same argument follows for g_{02}, g_{03}.

As a note, in equation (A) if we set $t = \text{constant}$, $r = \text{constant}$ then the line element associated with the surface of the sphere is

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

as $dr^2 = dt^2 = 0$.

We now state (without proof) that the line element corresponding to our assumption of

(a) That the field is static
(b) That the field is spherically symmetric
(c) That the spacetime is empty
(d) That the spacetime is asymptotically flat

is given by the line element

$$ds^2 = A(r)\, dt^2 - B(r)\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

To match the asymptotic behavior we require

$$A(r) \longrightarrow c^2 \quad \text{as} \quad r \longrightarrow \infty$$
$$B(r) \longrightarrow 1 \quad \text{as} \quad r \longrightarrow \infty$$

An important point: since $B(r)$ is not necessarily 1 we can only interpret r as being a <u>radial coordinate</u>, it is not necessarily a <u>radial distance</u>.

So now we have metric components in an $(x^0, x^1, x^2, x^3) \equiv (t, r, \theta, \phi)$ coordinate system. The non-zero components are

$$\begin{aligned}
g_{00} &= A(r) & &\Leftrightarrow & g^{00} &= 1/A(r) \\
g_{11} &= -B(r) & &\Leftrightarrow & g^{11} &= -1/B(r) \\
g_{22} &= -r^2 & &\Leftrightarrow & g^{22} &= -1/r^2 \\
g_{33} &= -r^2 \sin^2\theta & &\Leftrightarrow & g^{33} &= \frac{-1}{r^2 \sin^2\theta}
\end{aligned}$$

We now need to constrain $A(r)$ & $B(r)$ using the field equation

$$R_{\mu\nu} = 0$$

This is equivalent to

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\nu\sigma} = \partial_\nu \Gamma^{\sigma}{}_{\mu\sigma} - \partial_\sigma \Gamma^{\sigma}{}_{\mu\nu} + \Gamma^{\rho}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\rho\nu} - \Gamma^{\rho}{}_{\mu\nu}\Gamma^{\sigma}{}_{\rho\sigma} = 0 \qquad \text{(B)}$$

So if we calculate the component of the metric connection we can then constrain $A(r)$ & $B(r)$ (derivatives of them).

There are two ways of calculating the connection coefficient. One is directly from the equation of the metric connection, the other is via the Euler-Lagrange equation. In this case the metric connection method is simple enough for us to use.

Starting from

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right)$$

We find

$$\Gamma^{0}{}_{01} = \frac{1}{2} g^{0\delta}\left\{\partial_0 g_{\delta 1} + \partial_1 g_{0\delta} - \cancel{\partial_\delta g_{01}}\right\}$$

$$= \frac{1}{2} g^{00}\left\{\cancel{\partial_0 g_{01}} + \partial_1 g_{00}\right\}$$

$$= \frac{1}{2}\frac{1}{A}\left\{\partial_r A(r)\right\} = \frac{A'(r)}{2A(r)} \qquad ' \equiv \partial_r$$

$$\Gamma^1{}_{00} = \frac{1}{2} g^{1\delta} \left\{ \partial_0 g_{\delta 0} + \partial_0 g_{0\delta} - \partial_\delta g_{00} \right\}$$

$$= \frac{1}{2} g^{11} \left\{ \cancel{\partial_0 g_{10}} + \cancel{\partial_0 g_{01}} - \partial_1 g_{00} \right\}$$

$$= \frac{1}{2} \cdot -\frac{1}{B(r)} \left\{ -\frac{\partial}{\partial r} A(r) \right\} = \frac{A'(r)}{2B(r)}$$

$$\Gamma^1{}_{11} = \frac{1}{2} g^{1\delta} \left\{ \partial_1 g_{\delta 1} + \partial_1 g_{1\delta} - \partial_\delta g_{11} \right\}$$

$$= \frac{1}{2} g^{11} \left\{ \partial_1 g_{11} + \cancel{\partial_1 g_{11}} - \cancel{\partial_1 g_{11}} \right\}$$

$$= \frac{1}{2} \cdot -\frac{1}{B(r)} \cdot \left\{ \frac{\partial}{\partial r} - B(r) \right\} = \frac{B'(r)}{2B(r)}$$

$$\Gamma^1{}_{22} = \frac{1}{2} g^{1\delta} \left\{ \partial_2 g_{\delta 2} + \partial_2 g_{2\delta} - \partial_\delta g_{22} \right\}$$

$$= \frac{1}{2} g^{11} \left\{ \cancel{\partial_2 g_{12}} + \cancel{\partial_2 g_{21}} - \partial_1 g_{22} \right\}$$

$$= \frac{1}{2} \cdot -\frac{1}{B(r)} \cdot \left\{ -\frac{\partial}{\partial r} - r^2 \right\} = \frac{-r}{B(r)}$$

$$\Gamma^1{}_{33} = \frac{1}{2} g^{1\delta} \left\{ \partial_3 g_{\delta 3} + \partial_3 g_{3\delta} - \partial_\delta g_{33} \right\}$$

$$= \frac{1}{2} g^{11} \left\{ \cancel{\partial_3 g_{13}} + \cancel{\partial_3 g_{31}} - \partial_1 g_{33} \right\}$$

$$= \frac{1}{2} \cdot -\frac{1}{B(r)} \left\{ -\frac{\partial}{\partial r} \cdot -r^2 \sin^2\theta \right\} = \frac{-r \sin^2\theta}{B(r)}$$

$$\Gamma^1{}_{00} = \frac{1}{2} g^{1\delta} \{ \partial_0 g_{\delta 0} + \partial_0 g_{0\delta} - \partial_\delta g_{00} \}$$

$$= \frac{1}{2} g^{11} \{ \cancel{\partial_0 g_{10}} + \cancel{\partial_0 g_{01}} - \partial_1 g_{00} \}$$

$$= \frac{1}{2} \cdot -\frac{1}{B(r)} \left\{ -\frac{\partial}{\partial r} A(r) \right\} = \frac{A'(r)}{2B(r)}$$

$$\Gamma^1{}_{11} = \frac{1}{2} g^{1\delta} \{ \partial_1 g_{\delta 1} + \partial_1 g_{1\delta} - \partial_\delta g_{11} \}$$

$$= \frac{1}{2} g^{11} \{ \partial_1 g_{11} + \cancel{\partial_1 g_{11}} - \cancel{\partial_1 g_{11}} \}$$

$$= \frac{1}{2} \cdot -\frac{1}{B(r)} \cdot \left\{ \frac{\partial}{\partial r} -B(r) \right\} = \frac{B'(r)}{2B(r)}$$

$$\Gamma^1{}_{22} = \frac{1}{2} g^{1\delta} \{ \partial_2 g_{\delta 2} + \partial_2 g_{2\delta} - \partial_\delta g_{22} \}$$

$$= \frac{1}{2} g^{11} \{ \cancel{\partial_2 g_{12}} + \cancel{\partial_2 g_{21}} - \partial_1 g_{22} \}$$

$$= \frac{1}{2} \cdot -\frac{1}{B(r)} \cdot \left\{ -\frac{\partial}{\partial r} -r^2 \right\} = \frac{-r}{B(r)}$$

$$\Gamma^1{}_{33} = \frac{1}{2} g^{1\delta} \{ \partial_3 g_{\delta 3} + \partial_3 g_{3\delta} - \partial_\delta g_{33} \}$$

$$= \frac{1}{2} g^{11} \{ \cancel{\partial_3 g_{13}} + \cancel{\partial_3 g_{31}} - \partial_1 g_{33} \}$$

$$= \frac{1}{2} \cdot -\frac{1}{B(r)} \left\{ -\frac{\partial}{\partial r} \cdot -r^2 \sin^2\theta \right\} = \frac{-r \sin^2\theta}{B(r)}.$$

The remaining components of interest are

$$\Gamma^2{}_{12} = \frac{1}{r}, \quad \Gamma^2{}_{33} = -\sin\theta\cos\theta, \quad \Gamma^3{}_{13} = \frac{1}{r}, \quad \Gamma^3{}_{23} = \cot\theta$$

All other components not related via the symmetry of the lower 2 indices are zero.

Lots of tedious substitution in equation (B) gives

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} = 0 \qquad \text{——(1)}$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} = 0 \qquad \text{——(2)}$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 0 \qquad \text{——(3)}$$

$$R_{33} = R_{22}\sin^2\theta = 0 \qquad \text{——(4)}$$

The $R_{\mu\nu}$ for all $\mu \neq \nu$ vanish.

Equations (3) & (4) are essentially identical.
However

$$R_{00} \times \frac{B}{A} + R_{11} = 0$$

gives $\quad \dfrac{A'}{A} + \dfrac{B'}{B} = 0 \quad \Rightarrow \quad A'B + B'A = 0$ ——(C)

We can constrain further by recalling that as $r \to \infty$ $\quad A.B \to c^2.1$

Since $\quad A'B + B'A = 0 \quad \Rightarrow \quad AB = \text{constant}$

$\therefore \quad AB = c^2 \quad$ everywhere.

Thus

$$B = \frac{c^2}{A} \quad \text{and} \quad B' = \frac{-c^2 A'}{A^2} \quad \text{from (C)}$$

Substitution back into (3) gives

$$A - c^2 + \frac{Ar}{2}\left(\frac{2A'}{A}\right) = 0$$

$$\therefore \quad A + rA' = c^2$$

$$\Rightarrow \quad \frac{d}{dr}(rA) = c^2$$

$$\therefore \quad A(r) = c^2\left(1 + \frac{k}{r}\right)$$ where k is a constant of integration.

Where gives

$$B(r) = \left(1 + \frac{k}{r}\right)^{-1}$$

What does k represent? It must correspond (somehow) to the mass of the object producing the gravitational field. Although we have not yet studied the Newtonian limit of GR, it is found that

$$g_{00} \simeq 1 + \frac{2\phi}{c^2}$$

where ϕ is the usual Newtonian potential, $\phi = -\frac{Gm}{r}$
~~Assuming~~ We have already assumed space is asymptotically flat so we therefore identify the Newtonian r with our general relativistic r. Thus we find

$$k \simeq -\frac{2GM}{c^2}$$