PHYS4390 - Assignment 4

Due on Fri, Mar 16, 2012 (Time allowed=two weeks)

Show all working.

(1) (Example exam question from 2009.) This question looks at the metric in a rotating reference frame. The line element of a *flat spacetime*, in which we set $c = 1$, rotating with an angular velocity Ω about the z-axis of an inertial frame is,

$$ds^2 = [1 - \Omega^2(x^2 + y^2)]dt^2 - 2\Omega(ydx - xdy)dt - dx^2 - dy^2 - dz^2$$

(a) Verify that under the substitution $\phi \to \phi' = \phi - \Omega t$, this line element reduces to the Minkowski metric written in polar coordinates.
(b) Find the geodesic equations for x, y and z in the rotating frame.
(c) Consider a metric of the form, $g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta}$, where ϵ is small, $\eta_{\alpha\beta}$ is the metric of special relativity, and $h_{\alpha\beta}$ is another symmetric rank 2 tensor. The line element then becomes,

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta + \epsilon h_{\alpha\beta}dx^\alpha dx^\beta.$$

If a particle moves with speed v such that $v/c \sim \epsilon$ and $dx^i = v^i dt$, then, by considering the above form of the line element, and by subsituting for dx^i, show that in this metric,

$$\frac{dt}{d\tau} = 1 + \mathcal{O}(\epsilon) \quad \frac{d^2t}{d\tau^2} \simeq 0.$$

(d) Ignoring the terms of $\mathcal{O}(\epsilon)$, apply these results to to the rotating frame problem to derive the equation of motion for x in the non-relativistic limit. What do the three terms in the equation you derive correspond to?

What follows is a two-part question that gives an alternative derivation of the Schwarschild metric. We first need to show some auxiliary results that will be used in the derivation.
(2) (a) Suppose you are given a metric $g_{\mu\nu}$ such that $g_{\mu\nu} = 0$ if $\mu \neq \nu$. Show that for this particular metric the metric connection coefficients are given by,

$$\Gamma^\nu{}_{\mu\lambda} = 0$$

$$\Gamma^\mu{}_{\lambda\lambda} = -\frac{1}{2g_{\mu\mu}}\frac{\partial g_{\lambda\lambda}}{\partial x^\mu}$$

$$\Gamma^\mu{}_{\mu\lambda} = \frac{\partial}{\partial x^\lambda}\left(\ln\sqrt{|g_{\mu\mu}|}\right)$$

$$\Gamma^\mu{}_{\mu\mu} = \frac{\partial}{\partial x^\mu}\left(\ln\sqrt{|g_{\mu\mu}|}\right)$$

where $\mu \neq \nu \neq \lambda$ and there is *no summation* over repeated indices in these equations.
In looking for an isotropic and static metric we assume that the $g_{\alpha\beta}$ are independent of time and that $g_{0\beta} = 0$. Working in coordinates such that $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, a form of the line element (c=1) that meets these requirements is,

$$d\tau^2 = e^{2\nu(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2d\theta^2 - r^2\sin\theta^2 d\phi^2.$$

Note do not confuse the functions $\lambda(r)$ and $\nu(r)$ with their use as an index! This is an unfortunate convention for these particular functions.
(b) What are the values of the rank 2 covariant metric tensor $g_{\alpha\beta}$ for this line element? Also give the values of $g^{\alpha\beta}$.

(c) From these values, use the results of question 2 to show that (where prime denotes differentiation wrt r),

$$\Gamma^1_{\ 00} = \nu' e^{2(\nu-\lambda)}, \qquad \Gamma^0_{\ 10} = \nu'$$

$$\Gamma^1_{\ 11} = \lambda', \qquad \Gamma^2_{\ 12} = \Gamma^3_{\ 13} = \frac{1}{r},$$

$$\Gamma^1_{\ 22} = -re^{-2\lambda}, \qquad \Gamma^3_{\ 23} = \cot\theta,$$

$$\Gamma^1_{\ 33} = -r\sin^2\theta e^{-2\lambda}, \qquad \Gamma^2_{\ 33} = -\sin\theta\cos\theta.$$

(3) (a) Using the formula for the Ricci tensor,

$$R_{\mu\nu} = \partial_\nu \Gamma^\alpha_{\ \mu\alpha} - \partial_\alpha \Gamma^\alpha_{\ \mu\nu} - \Gamma^\alpha_{\ \mu\nu}\Gamma^\beta_{\ \alpha\beta} + \Gamma^\alpha_{\ \mu\beta}\Gamma^\beta_{\ \nu\alpha}$$

show using the results from question 2(c) that, for this metric,

$$R_{00} = (-\nu'' + \lambda'\nu' - (\nu')^2 - \frac{2\nu'}{r})e^{2(\nu-\lambda)},$$

$$R_{11} = \nu'' - \lambda'\nu' + (\nu')^2 - \frac{2\lambda'}{r},$$

$$R_{22} = (1 + r\nu' - r\lambda')e^{-2\lambda} - 1,$$

$$R_{33} = R_{22}\sin^2\theta.$$

(b) We showed in the lectures that since the field equations for the vacuum are $G_{\alpha\beta} = 0$, this also leads to $R_{\alpha\beta} = 0$. There is also a requirement that

$$\lambda + \nu = 0,$$

for the system to be asymptotically flat, use this plus the results of part (a) to show,

$$(1 + 2r\nu')e^{2\nu} = 1,$$

and then solve this to find ν (HINT: consider functions that have log(something) forms). From this result show that

$$g_{00} = 1 - \frac{A}{r}$$

where A is a constant of integration. Hence using the relationships between the ν and λ functions give the remaining terms in the metric tensor.

(4) This will be the first example of looking at a new geometry and then calculating its Ricci curvature directly. (a) Suppose you are given a 3-d Minkowski metric,

$$ds^2 = -dt^2 + dx^2 + dy^2$$

(a different metric signature has been introduced just for this question). Rewrite this in terms of the following coordinates,

$$x = r\cos\phi\sinh\eta$$

$$y = r\sin\phi\sinh\eta$$

$$t = r\cosh\eta$$

but we'll restrict to $dr = 0$ (we're actually embedding a surface into 2-d). Show that in the 2-d geometry the metric values are $g_{00} = r^2$, $g_{11} = r^2\sinh^2\eta$ and hence that the non-zero metric connection coefficients are,

$$\Gamma^\eta_{\ \phi\phi} = -\cosh\eta\sinh\eta, \quad \text{and,} \quad \Gamma^\phi_{\ \phi\eta} = \Gamma^\phi_{\ \eta\phi} = \coth\eta.$$

(b) For a 2-d system with a diagonal metric, the only value of Riemann tensor of interest is (no summation convention on η and ϕ in the following formula),

$$R_{\phi\eta\phi}^{\ \ \ \ \eta} = \partial_\eta \Gamma^\eta_{\ \phi\phi} - \partial_\phi \Gamma^\eta_{\ \phi\eta} + \Gamma^\alpha_{\ \phi\phi}\Gamma^\eta_{\ \eta\alpha} - \Gamma^\alpha_{\ \phi\eta}\Gamma^\eta_{\ \phi\alpha}.$$

Use this result to calculate the contractions of the Riemann tensor that give the Ricci tensor ($R_{ab} = R_{acb}^{\ \ \ \ c}$), and from that the contractions ($R = g^{ab}R_{ab}$) Ricci curvature scalar R, and hence that $R = -2/r^2$.