PHYS4390 - Assignment 3

Due in Friday, Mar 2, 2012 (Time allowed=two weeks)

Show all working.

(1) (a) Suppose a two dimensional surface is given by,

$$(x, y, z) = (\cos\theta(e + f\cos\phi), \sin\theta(e + f\cos\phi), e\sin\phi),$$

(here $e > f$ are just constants, $0 \leq \theta, \phi < 2\pi$) evaluate the (spatial, *i.e.* no time component) metric tensor on this surface.
(b) Do the same for a surface described by f and θ where,

$$(x, y, z) = (f\cos\theta, f\sin\theta, af),$$

here a is a constant, $f > 0$ is and $0 \leq \theta < 2\pi$.

(2) Suppose you are given the following coordinate transformation,

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan(y/x)$$

$$z = z$$

(a) Evaluate the (spatial) metric in (r, θ, z) coords.
(b) Calculate the connection coefficients, $\Gamma^{\alpha}{}_{\beta\gamma}$.

(3) Show that the metric connection,

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\frac{\partial g_{\delta\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}}\right),$$

satisfies the transformation relation for connection coefficients,

$$\Gamma'^{\alpha}{}_{\beta\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}\frac{\partial x^{\mu}}{\partial x'^{\gamma}}\Gamma^{\delta}{}_{\nu\mu} + \frac{\partial x'^{\alpha}}{\partial x^{\delta}}\frac{\partial^2 x^{\delta}}{\partial x'^{\beta}\partial x'^{\gamma}}$$

(HINT: be careful with your notation on the indices - this derivation is very lengthy, and is largely an exercise in careful writing, but it is not conceptually difficult.)

(4) (a) We showed that the metric connection preserves inner products under parallel transport. Using the definition of the square of a vector,

$$N^2 = g_{\alpha\beta}n^{\alpha}n^{\beta},$$

and by considering the absolute derivative of N^2 (in terms of connection coefficients), show that the metric connection ensures the length of n^{α} is preserved. (HINT: although it feels like a bit of a cheat, you can assume that $Dn^{\alpha}/Du = 0$.)
(b) Hence, using the definition of the angle between 4-vectors,

$$\cos(X, Y) = \frac{g_{ab}X^aY^b}{|g_{cd}X^cX^d|^{1/2}|g_{ef}Y^eY^f|^{1/2}}$$

show, along a geodesic, that the angle between a tangent vector and another parallel transported vector is preserved.

See overpage.

(5) Verify the symmetries of the Riemann tensor,

$$R_{abcd} = -R_{abdc},$$

$$R_{abcd} = -R_{bacd},$$

$$R_{abcd} = R_{cdab}.$$

(6) Suppose we are given an affine parameter λ. Recall the equation for such a geodesic equation along the path parameterized by λ is given by,

$$\nabla_X X^a = X^b \nabla_b X^a = 0$$

where X^a are the tangent vectors along the path $(dx^a/d\lambda)$.
(a) Show that this is equivalent to,

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0.$$

(b) Suppose now we have a new parameter $\mu = \mu(\lambda)$ which is another parameterization of the geodesic. If we define $W^a = dx^a/d\mu$ then show that for μ the geodesic equation becomes,

$$W^b \nabla_b W^a = f(\mu, \lambda) W^a,$$

and find the expression for the function $f(\mu, \lambda)$. (HINT: Be careful during the derivation, and remember that $W^a \partial_a = d/d\mu$.)
(c) For this new parameterization to have

$$W^b \nabla_b W^a = 0,$$

the function $f(\mu, \lambda)$ must be zero, which will define the expression for μ in terms of λ. Give the general solution for μ as a function of λ.