PHYS4390 - Assignment 2

Due in Class on Wed, Feb 15, 2012 (Time allowed=two weeks)

Show all working.

(1) (a) Is the following equation a legitimate tensor equation:

$$x^{\mu} = y_{\mu}?$$

Explain your answer.
(b) Write out the matrix equation $A = BC$ using index notation.
(c) By manipulating dummy indices find an alternative expression for

$$(A_{\mu\nu\gamma} + A_{\gamma\mu\nu} + A_{\nu\mu\gamma})M^{\mu}N^{\nu}P^{\gamma},$$

which only includes the tensor $A_{\mu\nu\gamma}$ multiplied by combinations of the other tensors.
(d) show that $\delta^b_a X^a = X^b$ and that $\delta^b_a X_b = X_a$.
(e) show that $\delta^b_a \delta^c_b \delta^d_c = \delta^d_a$.
(f) Write out the dot and cross products of two vectors using index notation (HINT: you will need more than just the vector components for one of these.)

(2) For a tensor, R_{ab}, and contravariant vector, X^a, verify or state whether the following are true/false
(a) (True/false) $X^a X^b = X^b X^a$
(b) (True/false) $R_{ab} = R_{ba}$
(c) (Verify) if R_{ab} is antisymmetric then $R_{aa} = 0$
(d) (Verify) if R_{ab} is symmetric then $R_{ab}Y^{ab} = R_{ab}Y^{(ab)}$
(e) (Verify) if R_{ab} is antisymmetric and Y^{ab} is symmetric then $R_{ab}Y^{ab} = 0$.
(f) You don't actually have to do this explicitly, but how would you go about showing the identity:

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

(g) (Show) The symmetric part of a tensor can actually be separated into two further distinct components, one component $\bar{T}^{ab}$ will have a trace of zero (i.e. the sum of the diagonal elements is zero), while the other component $\hat{T}^{ab}$ will be the identity (i.e. δ^{ab}) multiplied by something. (HINT: The fact that $\bar{T}^{ab}$ has zero trace will allow you determine what the factor that multiplies the identity is.)

(3) (a) Compute the Lie Brackets of the vector fields

$$\mathbf{V}_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

$$\mathbf{V}_2 = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$

$$\mathbf{V}_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

(b) Use the results you get to give a one line proof of the Jacobi identity for these vectors.

(4) Consider coordinate triplets which lie on a sphere of radius 1 centered on the origin of a Cartesian coordinate system. For these coordinates we will define charts that map the two dimensional surface to the Euclidean plane by,

$$M_N(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right),$$

$$M_S(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

(a) The maps are defined everywhere except two points - which ones?
(b) Since the charts overlap and map the unit sphere into the Euclidean plane, $M_N \circ M_S^{-1}$ on a point (u,v) must represent a smooth mapping of the plane from itself to itself (*i.e.* a map from (u,v) to (u',v')). (HINT: Calculate M_S^{-1} first by looking at how the pair (u,v) maps back into three dimensions.)

(5) Suppose that $B^{a_1 \ldots a_r}{}_{b_1 \ldots b_s}$ is a type(r,s) tensor field. Calculate how it's partial derivative (with respect to an index c) transforms under coordinate changes.

(6) (a) Show that the Lie derivative on functions and vector fields (so you'll need to consider two proofs) obeys,

$$[L_X, L_Y] = L_{[X,Y]},$$

for vector fields X & Y.
(b) Prove the Jacobi identity for Lie derivatives for three vector fields X,Y,Z (on both functions and vector fields).
(c) Show that the Lie derivative obeys,

$$L_X(fY) = (L_X f)Y + f(L_X Y),$$

for vector fields X and Y and a function f.

(7) Assuming that

$$\nabla_c X^a = \partial_c X^a + \Gamma^a{}_{bc} X^b$$

and that

$$\nabla_a \phi = \partial_a \phi,$$

where ϕ is a scalar, show that for the covariant derviative to be Liebniz then

$$\nabla_c X_a = \partial_c X_a - \Gamma^b{}_{ac} X_b.$$

(8) Show that the connection coefficients $\Gamma^a{}_{bc}$ are not tensors.