Goldstein Chapter 4.

Rigid Body Kinematics

Kinematics: analysis of the nature and characteristics of motion. Consequently, kinematics is not concerned with variations due to forces, this is described via dynamics.

Coordinate description of a Rigid Body

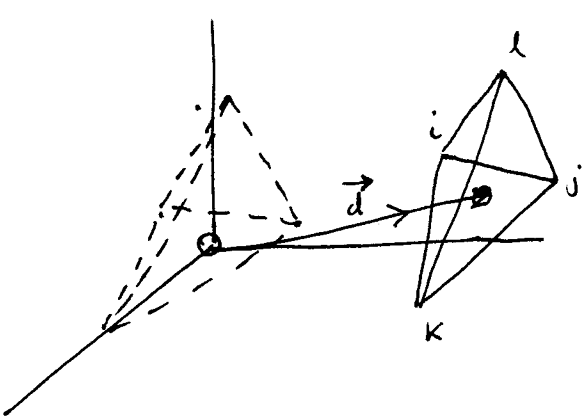
Intuition suggests that 6 coordinates should be sufficient to specify a rigid body's configuration:

(1) 3 coordinates to describe a displacement
(2) " " " " a relative orientation

Recall that for rigid bodies we have a series of constraints describing the distances between parts:

$$\left(\vec{r}_i - \vec{r}_j\right)^2 = c_{ij}^2 \equiv r_{ij}^2 \qquad \left(\text{ie} \quad r_{ij} = \left|\vec{r}_i - \vec{r}_j\right|\right)$$

Let's investigate the configuration of a body with 4 corners.



The translation fixes the position of one corner immediately (let's choose i). We've used the 3 displacement degrees of freedom to do this & they are contained in the vector $\vec{d}$

The orientation of the body can be further described by considering the locus of points associated with corners away from i. Let's take j.

We know

$$r_{ij} = c_{ij}$$

Hence, the possible positions for j are described by the surface of a sphere - just think of rotating the object while keeping corner i fixed.

Thus, fixing j uses 2 degrees of freedom.

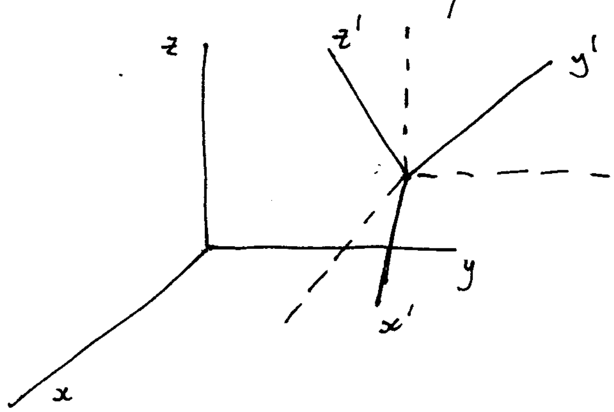
Once i & j are fixed k & l are specified by a rotation - the final degree of freedom.

Thus these three degrees of freedom (ie generalized coordinates) plus the three from the displacement confirms our intuition.

Description of the Orientation

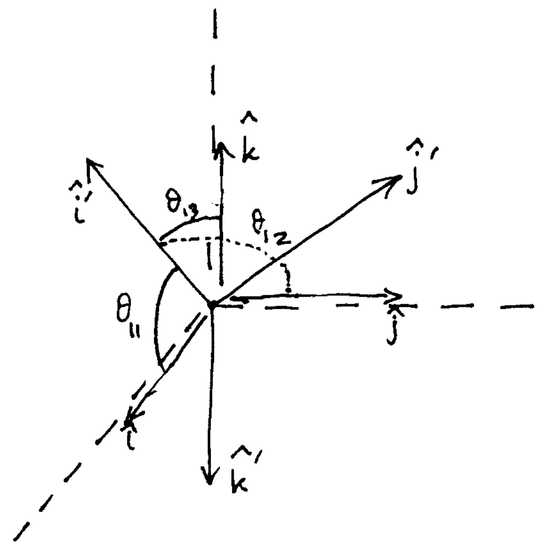
Let's consider a set of coordinates x', y', z' set at the origin of the body oriented in a preset direction within the body (it doesn't matter precisely what the orientation is). (x', y', z' rotate with the body)

An external reference set of coordinates is x, y, z



Let the unit vectors for x, y, z be $\hat{i}, \hat{j}, \hat{k}$,
" " " " " x', y', z' " $\hat{i}', \hat{j}', \hat{k}'$

The relative displacement of the axes does not impact the rotation between the two sets of axes. How do we describe x', y', z' in terms of x, y, z?



Between each pair of primed & unprimed axes we have angle θ. Using dot products:

$$\hat{i}' \cdot \hat{i} = \cos \theta_{11}$$

Where 1 denotes i and the first index corresponds to the $'$ coord.

e.g. $\hat{j}' \cdot \hat{j} = \cos \theta_{22}$
$\hat{k}' \cdot \hat{k} = \cos \theta_{33}$

We also have

$$\hat{i}' \cdot \hat{j} = \cos \theta_{12} \quad \text{but} \quad \hat{j}' \cdot \hat{i} = \cos \theta_{21} \neq \cos \theta_{12}$$

similarly

$$\hat{i}' \cdot \hat{k} = \cos \theta_{13} \neq \hat{i} \cdot \hat{k}' = \cos \theta_{31}$$

We can use these angles to resolve components of $\hat{i}', \hat{j}', \hat{k}'$ in terms of $\hat{i}, \hat{j}, \hat{k}$

$$\left.\begin{aligned} \hat{i}' &= \cos \theta_{11} \hat{i} + \cos \theta_{12} \hat{j} + \cos \theta_{13} \hat{k} \\ \hat{j}' &= \cos \theta_{21} \hat{i} + \cos \theta_{22} \hat{j} + \cos \theta_{23} \hat{k} \\ \hat{k}' &= \cos \theta_{31} \hat{i} + \cos \theta_{32} \hat{j} + \cos \theta_{33} \hat{k} \end{aligned}\right\} \text{— (A)}$$

If we have a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then the components of $\vec{r}$ in the x', y', z' coordinates can be found by:

$$x' = \vec{r}\cdot\hat{i}' = x\,\hat{i}\cdot\hat{i}' + y\,\hat{j}\cdot\hat{i}' + z\,\hat{k}\cdot\hat{i}'$$
$$= x\cos\theta_{11} + y\cos\theta_{12} + z\cos\theta_{13}$$

and

$$y' = \cos\theta_{21}\,x + \cos\theta_{22}\,y + \cos\theta_{23}\,z$$
$$z' = \cos\theta_{31}\,x + \cos\theta_{32}\,y + \cos\theta_{33}\,z$$

The procedure generalizes to any vector, e.g. $\vec{G} = (G_x, G_y, G_z)$ then

$$G_x' = \cos\theta_{11}\,G_x + \cos\theta_{12}\,G_y + \cos\theta_{13}\,G_z$$

Thus these nine direction cosines describe the relative orientation & mapping between the two coordinate systems. Note this orientation may change with time.

However, these 9 angles are not independent (since we've argued you only need 3 degrees of freedom to specify the orientation).

We can find a relationship between the direction cosines by using the fact that the unit vectors are orthogonal.

i.e. $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Same applies to $\hat{i}', \hat{j}', \hat{k}'$!

Then using the expansions for $\hat{i}', \hat{j}', \hat{k}'$ in (A) we can construct $\hat{i}'.\hat{i}'$, $\hat{i}'.\hat{j}'$ etc in terms of $\hat{i}, \hat{j}, \hat{k}$.

e.g. $$\hat{i}'.\hat{i}' = (\cos\theta_{11}\,\hat{i} + \cos\theta_{12}\,\hat{j} + \cos\theta_{13}\,\hat{k})^2 = 1$$

$$\therefore \quad \cos^2\theta_{11} + \cos^2\theta_{12} + \cos^2\theta_{13} = 1$$

Considering all possible combinations of $\hat{i}, \hat{j}, \hat{k}$ in terms of $\hat{i}', \hat{j}', \hat{k}'$

$$\sum_{l=1}^{3} \cos\theta_{ln}\cos\theta_{lm} = 0 \qquad n \neq m$$

$$\sum_{l=1}^{3} \cos^2\theta_{lm} = 1$$

These six equations reduce the number of degrees of freedom from 9 to 3. Note that in terms of the kronecker delta the above equations can be written

$$\sum_{l=1}^{3} \cos\theta_{ln}\cos\theta_{lm} = \delta_{nm} \qquad \text{——— (B)}$$

The three degrees of freedom describing the rotation are called the Euler angles. Before examining them in more detail we first consider so called 'orthogonal' transformations in more detail.

The transformation we have been discussing can be written in matrix form. Let $x \to x_1$, $y \to x_2$, $z \to x_3$ & let $\cos\theta_{ij} = a_{ij}$. Then

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

i.e. $\vec{x}\,' = A\vec{x}$ where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The orthogonality of the basis vectors $\hat{i}, \hat{j}, \hat{k}$ and the primed versions lead to the "orthogonality condition"

$$\sum_{l=1}^{3} \cos\theta_{ln} \cos\theta_{lm} = \delta_{nm}$$

which can be written

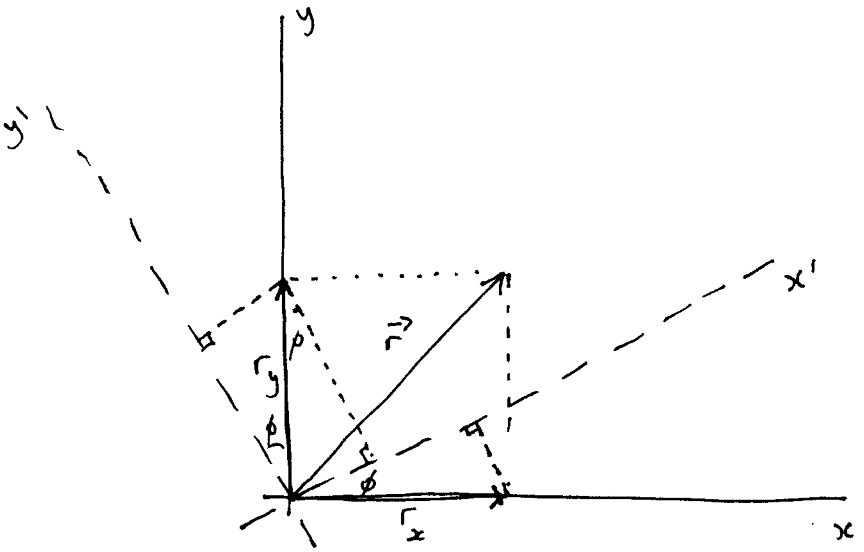
$$\sum_{l=1}^{3} a_{ln}\, a_{lm} = \delta_{nm} \qquad \text{——} (c)$$

Any matrix with elements obeying the relation (c) is said to represent an orthogonal transformation.

Rotation in the x-y plane (i.e. about the z axis) is described by a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can describe these a_{ij} in terms of a single rotation angle ϕ



Resolve components of r_x on to x' axis

$$r_x' = r_x \cos\phi + r_y \sin\phi$$

For r_y' we find

$$r_y' = r_y \cos\phi - r_x \sin\phi$$
$$r_z' = r_z$$

Hence the matrix form is

$$A = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly for a rotation about the y & x-axes we have

$$A = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \qquad (y\text{-axis})$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \qquad (x\text{-axis})$$

Interpretation of Matrix-Vector multiplication

There are two ways of interpreting the operation of the matrix A on $\vec{r}$. We have talked about

$$\vec{r}\,' = A\vec{r}$$

where $\vec{r}\,'$ are the new components of the vector in the primed coordinate system. Goldstein likes to emphasize that the $\vec{r}$ vectors on both sides are equivalent by writing

$$(\vec{r})' = A\vec{r}$$

The alternative view point is that this describes an "active" transformation of the vector $\vec{r}$, where

$$\vec{r}\,' = A\vec{r}$$

describes a new vector in the original coordinate system.

The origin of these two interpretations is related to the fact that A acts only on components of a vector, and not on the basis vectors as well. We can think of these two interpretations as

$$(\vec{r})' = A\vec{r}$$

components in primed coords — $(\vec{r})'$; components in unprimed coords — $\vec{r}$

<u>or</u>

$$\vec{r}\,' = A\vec{r}$$

new components in unprimed coords — $\vec{r}\,'$; components in unprimed coords — $\vec{r}$

Thus the difference is really whether we consider

$$(\vec{r})' = x'\hat{i}' + y'\hat{j}' + z'\hat{k}' \qquad \text{"passive"}$$

or

$$\vec{r}\,' = x'\hat{i} + y'\hat{j} + z'\hat{k} \qquad \text{"active"}$$

In the case of the $(\vec{r})'$ definition the change of the basis vectors can be viewed as offsetting the change in the components x', y', z'.

(If you multiply out all the $\hat{i}'$ in terms of $\hat{i}, \hat{j}, \hat{k}$ you will find $(\vec{r})' = \vec{r}$)

Properties of Orthogonal Transformation Matrices

Let's introduce the Einstein summation convention. Wherever repeated indices occur it is understood that they are summed over,

$$\text{ie} \quad \sum_{i=1}^{n} x_i x_i \equiv x_i x_i$$

To avoid ambiguity, if an index sum is not implied we will explicitly state this.

Useful results to note:

No summation convention on LHS here

$$\sum_{i=1}^{n} \delta_{ii} \equiv \delta_{ii} = n \quad \text{since } \delta_{mm} = 1 \text{ (no summation conv)}$$

$$\sum_{j=1}^{n} \delta_{ij}\delta_{jk} \equiv \delta_{ij}\delta_{jk} = \delta_{ik}$$

$$\sum_{j=1}^{n} \delta_{ij}x_j \equiv \delta_{ij}x_j = x_i$$

$$\sum_{i=1}^{n} x_i x_i = \sum_{j=1}^{n} x_j x_j = x_i x_i = x_j x_j$$

Notice we can re-label any summed index

$$\text{ie} \quad \delta_{ij}x_j = \delta_{ik}x_k$$

Under the summation convention we can express matrix multiplication as follows:

$$x'_k = A_{kj}\, x_j \qquad k = 1, 2, 3$$

j is summed from $1, \ldots, 3$

A_{kj} are the elements of the matrix.

Two consecutive multiplications would then be

$$x''_k = B_{kj} A_{jl} x_l = C_{kl} x_l$$

where $\quad C_{kl} = B_{kj} A_{jl}$

Note that orthogonal transformations are not in general commutative. ie matrices representing A & B have $AB \neq BA$, ie $B_{kj} A_{jl} \neq B_{lj} A_{jk}$
Orthogonal transformations are of course associative
$(AB)C = A(BC)$.

Further, given two orthogonal transformations $A B$,
$C = AB$
is also an orthogonal transformation. (Proof - homework).

We define in inverse of A by the operation

$$x_k = A'_{kj} x'_j$$ where A'_{kj} are elements of A^{-1}.

but since $x'_k = A_{kj} x_j$ we get

$$x'_k = A_{kj} A'_{jl} x'_l$$

$$\therefore \quad A_{kj} A'_{jl} = \delta_{kl} \quad \text{———} (1)$$

so δ_{kl} represents the identity matrix.

multiplying with A'_{ik} (including sum) we get

$$(A'_{ik} A_{kj}) A'_{jl} = A'_{ik} \delta_{kl} = A'_{il}$$

$$\delta_{ij} \; A'_{jl} = A_{il} \qquad \therefore \; A'_{ik} A_{kj} = \delta_{ij}$$

So (unsurprisingly) we've shown

$$A A' = A' A = A^{-1} A = A A^{-1} = \mathbb{1}$$

An additional important result is found if we consider the fact that orthogonal transformations must preserve the size of the transformed vector, ie

$$x_i' x_i' = x_i x_i$$

in terms of mapped x_i we get

make sure you use different summation indices

$$x_i x_i = A_{ij} x_j A_{ik} x_k$$

$$x_k x_k = A_{ij} A_{ik} x_j x_k$$

$$\delta_{kj} x_j x_k = A_{ij} A_{ik} x_j x_k$$

$$\therefore \quad A_{ij} A_{ik} = \delta_{jk}$$

If we define the transpose of A, A^T by $A^T_{ik} = A_{ki}$, then

$$A^T_{ji} A_{ik} = \delta_{jk}$$

can also show $\quad A_{ki} A^T_{ij} = \delta_{jk}$

$\therefore \quad A^T \equiv A^{-1}$ ie the inverse of an orthogonal matrix is the transpose (& vice versa).

What about the determinant? What values are allowed?

Since $\det(AB) = \det(A)\det(B)$

& $\det(A^T) = \det(A)$

$\det(A^T A) = \det(A^T)\det(A) = \det(\mathbb{1})$

$\Rightarrow \quad [\det(A)]^2 = 1$

$\therefore \quad \det(A) = \pm 1.$

Orthogonal transformations are thus broken into two types:

$\det(A) = 1 \quad \Rightarrow$ proper orthogonal transformation

$\det(A) = -1 \quad \Rightarrow$ improper orthogonal transformation

Improper orthogonal transformations include reflections & combinations of reflections & rotations.

Proper orthogonal transformations include <u>only</u> rotations. Taken together, the set of all possible rotations in 3d forms a mathematical object called a group, that is known as SO(3) - special orthogonal group of dimension 3.