Goldstein Chapter 2

Hamilton's Principle

D'Alembert's Principle allowed us to derive the E-L equation from a differential equation (a "differential principle"). Hamilton's Principle derives the same equation but begins from the variation of an integral.

Useful definition:

If a system is described by n generalized coordinates $q_1, \dots, q_n$, then the n-dimensional space represented by these coordinates is called configuration space.

As a system evolves over time it will thus follow a single path in configuration space. The path will be parameterized by time.

Hamilton's Principle:

The motion of the system from time t_1 to t_2 is such that the path integral (called the action or action integral)

$$\underline{I} = \int_{t_1}^{t_2} L(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t)\, dt \quad \text{—— (1)}$$

where $L = T - V$, has a stationary value for the actual path of motion.

This is equivalent to finding the path the extremizes the integral, so that

$$\delta I = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = 0$$

where q_i denotes all q va
$\dot{q}_i$ " " $\dot{q}$ "

Via the calculus of variations (see Goldstein 2.3 for the full derivation) this will lead directly to the n equations of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

This derivation assumes that the q_i are independent, such as we have discussed previously. However, "occasionally" it may not be true and instead we may still have to consider relationships between the q_i. Specifically, this happens when we consider constraints and are unable to write down a Lagrangian in terms of the true degrees of freedom.

Let us consider the case where the q_i are not independent & are connected via m constraint equations of the form

$$f_j(q_1, \ldots, q_n) = 0 \qquad j = 1, \ldots, m$$

Suppose we now consider a variation in the q_i

namely, δq_i, Then the variation in the f_j is

$$\delta f_j(q_1, \ldots, q_n, t) = \sum_i \frac{\partial f_j}{\partial q_i} \delta q_i = 0 \quad \text{— (3)}$$

We interpret this result as telling us the constraints force the δq_i into specific regions or "subspaces" of configuration space. Since there may be more than one of the f_j the δq_i must stay in the intersection of the subspaces.

Example

Suppose we have a constraint

$x^2 + y^2 = 1 \quad \Rightarrow \quad f(x, y) = x^2 + y^2 - 1 = 0$

Under variations of x & y we find f obeys

$$\frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y = 0$$

Thus $\quad x\,\delta x + y\,\delta y = 0$

which can be written as the vector equation

$$\vec{n} \cdot \delta \vec{r} = 0$$

where $\vec{n}$ is the normal to the "surface" described by $f(x,y)$ (in this case the "surface" is the circle of unit radius).

Remember: For a "surface" $f(q_1, \ldots, q_n)$ ∇f where $\nabla \equiv \hat{e}_i \frac{\partial}{\partial q_i}$ describes the vector normal to the surface.

This is the n-dimensional analog of the usual 3-d result.

The equation of the tangent "plane" will be

$$\frac{\partial f}{\partial q_1}(q_1 - q_1^\circ) + \dots + \frac{\partial f}{\partial q_n}(q_n - q_n^\circ) = 0$$

where $(q_1^\circ, \dots, q_n^\circ)$ is the point at which the normal is taken.

Hence $\delta f_j(q_1, \dots, q_n, t) = \sum_i \frac{\partial f_j}{\partial q_i} \delta q_i = 0$

is the equation of an infinitesimal plane or "subspace" in configuration space on which the δq_i must move to satisfy the constraint. (This equation describes what is called a tangent space).

How does this impact the calculus of variations approach to deriving the E-L equation

We again derive

$$\delta I = \int \left[\sum_i \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \right\} \delta q_i \right] dt = 0$$

but previously we argued that the δq_i were independent and that therefore we must have

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

However, we are now not able to make this assumption of the dq_i being independent.

We still require

$$\sum_i \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \right\} \delta q_i = 0$$

This is definitely true for systems with holonomic constraints since the absence of terms in $\dot{q}$ means we can assume the q_i are independent from one time to another.

This is the same form of equation as (B).

$$\sum_i \frac{\partial f_j}{\partial q_i} \delta q_i = 0$$

We must therefore have that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \propto \frac{\partial f_j}{\partial q_i} \qquad \text{for all } j \,\&\, i$$

$$\Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \nu_j \frac{\partial f_j}{\partial q_i}$$

where the ν_j are a set of unknown functions of t. Further, since this is true for all j we must have that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \sum_j \lambda_j \frac{\partial f_j}{\partial q_i}$$

Where the λ_j are known as <u>Lagrange Multipliers</u>. ($\nu_j \to \lambda_j$ is just a change of notation)

Thus when the q_i are subject to constraints we must solve

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_j \lambda_j \frac{\partial f_j}{\partial q_i} = 0$$

with the constraint $f_j(q_1, \dots, q_n) = 0$

Although we have not shown it, this method will also work for constraints of the form

$$f_j(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n; t) = 0.$$

Example

We've considered a simple pendulum with polar coordinates previously. We now consider the cartesian description with a constraint.

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$V = -mgy$$

Constraint is $f(x,y) = x^2 + y^2 - l^2 = 0$

Thus

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y$$

The E-L equations with constraints become

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \lambda \frac{\partial f}{\partial x} = 0$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \lambda \frac{\partial f}{\partial y} = 0$$

Which gives

$$-\frac{d}{dt}(m\dot{x}) - \lambda 2x = 0 \quad \Rightarrow \quad m\ddot{x} - \tilde{\lambda}x = 0 \quad \text{—}$$

$$mg - \frac{d}{dt}(m\dot{y}) - \lambda 2y = 0 \quad \Rightarrow \quad m\ddot{y} - mg - \tilde{\lambda}y = 0 \quad \text{— (}$$

$\tilde{\lambda} = 2\lambda$

We can get a third equation by taking the time derivative of the constraint $f(x,y)$

$$\Rightarrow \quad x\dot{x} + y\dot{y} = 0$$

$$\Rightarrow \quad \dot{x}^2 + \dot{y}^2 + x\ddot{x} + y\ddot{y} = 0 \quad \text{— (3)}$$

There are thus 3 equations & 3 unknowns.

The typical way to proceed would be either to solve all 3 equations, or $\tilde{\lambda}$ can be eliminated using eqns (1) & (2) to get

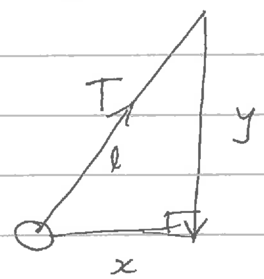
$$y\ddot{x} - x\ddot{y} + xg = 0$$

$$x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = 0$$

What do the λ mean physically? We could re-write eqns (1) & (2) as

$$m\ddot{x} - \hat{\lambda}\frac{x}{l} = 0$$

$$m\ddot{y} - mg - \frac{\hat{\lambda}}{l}y = 0$$



and $\hat{\lambda}$ is equivalent to the tension in the pendulum, ie a <u>constraint force</u>.

Hence the Lagrangian is

$$L = T - V$$
$$= \frac{1}{2} M(\dot{x}^2 + r^2\dot{\theta}^2) - Mg(l-x)\sin\phi$$

One constraint $\Rightarrow$ one Lagrange Multiplier, λ

$f = r\theta - x = 0$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = -1 \qquad \frac{\partial f}{\partial \theta} = r$$

E-L equations with constraint are

$$M\ddot{x} - Mg\sin\phi + \lambda = 0 \qquad \text{——} (1)$$

$$Mr^2\ddot{\theta} - \lambda r = 0 \qquad \text{——} (2)$$

We can differentiate the constraint to get

$$r\ddot{\theta} = \ddot{x}$$

Substituting this into (2) gives $M\ddot{x} = \lambda$

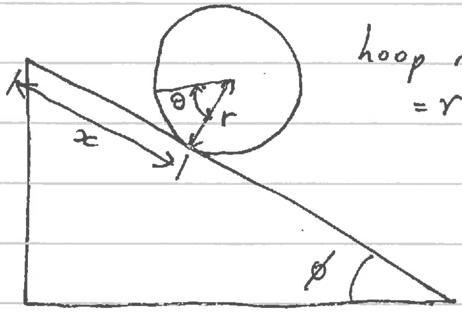
Substituting this result into (1) gives

$$\ddot{x} = \frac{g\sin\phi}{2}$$

$$\therefore \quad \lambda = \frac{Mg\sin\phi}{2} \quad \Rightarrow \ddot{\theta} = \frac{g\sin\phi}{2r}$$

So the rolling hoop has only half the acceleration of a block of mass M sliding without friction.

Example



hoop radius $= r$

Let's consider a hoop of mass M rolling down a plane with angle ϕ.

Position down the plane is x the 'hoop angle' θ describes its relative rotation.

Our constraint is that $x = r\theta + \delta$
Lets set the constant δ to zero by putting the ~~hoop~~ at $x = 0$ when $\theta = 0$.

$$\Rightarrow \quad f(\theta, x) = r\theta - x \quad \text{is the constraint}$$

Kinetic energy = energy due to centre of mass moving
+ " " to rotation of hoop

Centre of mass KE $= \frac{1}{2}Mv^2 = \frac{1}{2}M\dot{x}^2$

Rotation : $r\dot{\theta}$ = velocity of edge of hoop

$\therefore$ Rotational KE $= \frac{1}{2}Mr^2\dot{\theta}^2$

$$\therefore \quad T = \frac{1}{2}M(\dot{x}^2 + r^2\dot{\theta}^2)$$

For P.E., assume length of slope is l.
Normalize so that when $l = x$ $V = 0$

$$\therefore \quad V = Mg(l - x)\sin\phi$$

Goldstein 2.6.

Conservation Theorems

The E-L equations gives us a way of deriving the equations of a motion for a system. There will be n equations for a system with n degrees of freedom.
Since the equations of motion are second order there will be $2n$ constants of integration. These will be specified by the initial q_i's & $\dot{q}_i$'s (ie $2n$ total variables).

However, in the majority of cases, it will not be possible to directly integrate (solve) the equations of motion! However, that does not mean we cannot learn a great deal about the system.

As an example, consider particles moving in the absence of a potential.

$$L = T - V = \frac{1}{2} \sum_j m_j v_j^2$$

$$= \frac{1}{2} \sum_j m_j \left(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2 \right)$$

The only non-zero derivatives are

$$\frac{\partial L}{\partial \dot{x}_j} = m_j \dot{x}_j \qquad \frac{\partial L}{\partial \dot{y}_j} = m_j \dot{y}_j \qquad \frac{\partial L}{\partial \dot{z}_j} = m_j \dot{z}_j$$

If there are n particles the $3n$ equations of motion will be given by

$$\frac{\partial L}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = 0$$
$$\frac{\partial L}{\partial y_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_j}\right) = 0$$
$$\frac{\partial L}{\partial z_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_j}\right) = 0$$

Since L does not depend on x_j, y_j, z_j we get

$$\frac{d}{dt}(m_j \dot{x}_j) = 0 \qquad \frac{d}{dt}(m_j \dot{y}_j) = 0 \qquad \frac{d}{dt}(m_j \dot{z}_j) = 0$$

These equations can be combined to give n equations

$$\frac{d}{dt}(m\vec{v}_j) = \vec{0}$$

ie they represent equations for the conservation of momentum.

Definition:
The generalized momentum associated with coordinate q_j is defined by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

These may be called canonical/conjugate momenta. As for any generalized quantity they may not necessarily have units of momentum.

The free particle example where the Lagrangian does not depend on positions leads us to the following observation:

If a Lagrangian does not depend explicitly on the coordinate q_i (but may contain $\dot{q}_i$) then the coordinate is said to be "cyclic" or "ignorable".

For such a system the equation of motion reduces to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\therefore \quad \frac{d}{dt}(p_i) = 0$$

$$\Rightarrow \quad p_i = \text{constant}$$

Hence, the generalized momentum conjugate to a cyclic coordinate is conserved.

<u>Example</u> Particle moving in a central potential.
Consider a potential that is a function of r only. Use plane polar coords for simplicity.

$$L = \frac{1}{2} m v^2 - U(r)$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)$$

L has no dependence on θ, therefore

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

Since $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ and $\frac{\partial L}{\partial \theta} = 0$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

$$\therefore \quad mr^2\dot{\theta} = \text{constant} = l$$

$mr^2\dot{\theta}$ is the angular momentum of the system.

For the radial coordinate

$$\frac{\partial L}{\partial r} = -\frac{\partial U}{\partial r} + mr\dot{\theta}^2 \quad \& \quad \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\therefore \quad m\ddot{r} = -\frac{dU}{dr} + mr\dot{\theta}^2$$ radial force is the derivative of the P.E.

+ something like centrifugal

We can also show conservation of energy for this system.

Since $E = T + V = 2T - L$

$$\Rightarrow \quad \frac{dE}{dt} = 2\frac{dT}{dt} - \frac{dL}{dt}$$

$$= 2\frac{dT}{dt} - \cancel{\frac{\partial L}{\partial t}} - \frac{\partial L}{\partial r}\dot{r} - \cancel{\frac{\partial L}{\partial \theta}\dot{\theta}} - \frac{\partial L}{\partial \dot{r}}\ddot{r} - \frac{\partial L}{\partial \dot{\theta}}\dot{\theta}$$

$$= 2\frac{dT}{dt} - \frac{\partial L}{\partial r}\dot{r} - \frac{\partial L}{\partial \dot{r}}\ddot{r} - \frac{\partial L}{\partial \dot{\theta}}\ddot{\theta}$$

Since $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\dot{\theta}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)\dot{\theta} + \frac{\partial L}{\partial \dot{\theta}}\ddot{\theta}$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{\theta}}\ddot{\theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\dot{\theta}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)\dot{\theta}$$

and similarly

$$\frac{\partial L}{\partial \dot r}\ddot r = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\dot r\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right)\dot r$$

Substitute these results back, then

$$\frac{dE}{dt} = \frac{d}{dt}\left\{2T - \frac{\partial L}{\partial \dot r}\dot r - \frac{\partial L}{\partial \dot\theta}\dot\theta\right\} + \left\{\dot r\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) + \dot\theta\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \dot r\frac{\partial L}{\partial r}\right\} \quad \text{(A)}$$

However for the E-L equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{\partial L}{\partial r}$$

Thus the term in the second bracket in (A) is zero.

$$\therefore \quad \frac{dE}{dt} = \frac{d}{dt}\left\{2T - \frac{\partial L}{\partial \dot r}\dot r - \frac{\partial L}{\partial \dot\theta}\dot\theta\right\}$$

but

$$\frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta$$

$$\frac{\partial L}{\partial \dot r} = m\dot r$$

$$\therefore \quad \frac{dE}{dt} = \frac{d}{dt}\left\{2T - m\dot r^2 - mr^2\dot\theta^2\right\}$$

$$= 0$$