Goldstein 1.4.

D'Alembert's Principle

Consider first a system at equilibrium. In this case the sum of all the forces on a given particle i is zero. Thus for the particle i, $\vec{F}_i = \vec{0}$. This $\vec{F}_i$ will include both applied (ie dynamical) forces & constraint forces.

Let us now consider the work done at a time t when we change the coordinates of the system by an amount $\delta\vec{r}_i$. Note that we will call this displacement "virtual" to distinguish it from the changes that happen during dynamical evolution in a time δt. Imagine we are just perturbing the system very slightly.

If $\vec{F}_i = \vec{0}$, then the sum over all particles i dotted with the displacement vector $\delta\vec{r}_i$ must be zero:

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0 \qquad \text{—— (A)}$$

Since $\vec{F}\cdot\vec{r}$ corresponds to work, we say that the "virtual work" has vanished.

Let us now be more specific about $\vec{F}_i$. The underlying goal of our derivation will be to write down equations of motion in generalized coordinates where the constraint forces vanish.

Thus for $\vec{F}_i$ write

$$\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$$

where $\vec{F}_i^{(a)}$ is the "applied" force & $\vec{f}_i$ is the force of the constraints. We can then expand (A) to give

$$\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0$$

The second term is the work done by the constraint forces under the coordinate change. If we now restrict ourselves to considering systems for which

$$\sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0$$

Then we are considering only systems for which the net virtual work of the constraint is zero. This is true for a great number of constraints. e.g. Consider a particle constrained to move on a surface, the constraint force is perpendicular to the surface while the displacement will be along the surface. Similarly for a pendulum or rigid body.

Thus we can write the equilibrium condition as

$$\sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0 \qquad \text{—— (B)}$$

Note that now the $\vec{F}_i^{(a)}$ will in general be non-zero, as will the $\delta\vec{r}_i$. The $\delta\vec{r}_i$ will also be related to one another via the constraint equations.

Equation (B) is known as the "Principle of Virtual Work." It applies only to <u>static</u> situations though. We need a principle that can be applied at all times. Bernoulli & D'Alembert outlined the required procedure:

Since for any system

$$\vec{F}_i = \dot{\vec{p}}_i$$

we can write

$$\vec{F}_i - \dot{\vec{p}}_i = \vec{0}$$

Thus if we imagined that every particle in the system were given a "reversed effective force" $-\dot{\vec{p}}_i$, then the system would be in equilibrium. In this case, for a displacement $\delta\vec{r}_i$ we get

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0$$

If we again separate $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$ (ie into applied & constraint forces) then

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0$$

As before, we restrict ourselves to systems for which the net virtual work of the constraint is zero. Thus,

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0 \quad \text{———} (C)$$

which is known as <u>D'Alemberts' Principle</u>. Notice the forces of constraint do not appear.

However, we <u>cannot</u> argue that each of the components of $\vec{F}_i^{(a)} - \dot{\vec{p}}_i$ must be zero. All the $\delta\vec{r}_i$ are connected by the constraint equations (ie they won't all be independent). To be able to set all the components of $\vec{F}_i^{(a)} - \dot{\vec{p}}_i$ to zero we need the $\delta\vec{r}_i$ to all be independent. This implies we should transform the equation into <u>generalized coordinates</u>.

Starting from
$$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_n, t)$$
the Chain Rule gives

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

There is <u>no variation with respect to t</u> though. Virtual displacements consider only variations in the spatial coordinates.

We can now expand the terms in (C)

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_i \vec{F}_i \cdot \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$
$$= \sum_{i,j} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$
$$= \sum_j \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$
$$= \sum_j Q_j \delta q_j$$

we dropped the $^{(a)}$ from $\vec{F}_i^{(a)}$ since constraint forces no longer appear.

Where $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$ are known as the components of the <u>generalized force</u>.

Note that generalized coordinates need not have dimensions of length (e.g. they could be angles). Similarly, the Q_j need not necessarily have units of force (if the generalized coordinates were angles the Q_j would be torques).

Our next step is to examine the implication of the second term in D'Alembert's Principle (C).

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i$$

$$= \sum_i m_i \ddot{\vec{r}}_i \cdot \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$= \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

assuming mass constant in time.

Since

$$\sum_i \frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) = \sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} + \sum_i m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

$$\Rightarrow \sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left\{ \frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right\} \quad (F)$$

While $\dot{\vec{r}}_i = \vec{v}_i$ by definition, it is not immediately obvious what $\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$ is

Since $$\frac{d}{dt} \equiv \sum_k \frac{dq_k}{dt}\frac{\partial}{\partial q_k} + \frac{\partial}{\partial t}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \sum_k \dot{q}_k \frac{\partial^2 \vec{r}_i}{\partial q_k \partial q_j} + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

If we now assume that our function r is well behaved (ie smooth), then we can swap the order of the partial differentials and we find

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left\{ \sum_k \dot{q}_k \frac{\partial \vec{r}_i}{\partial q_k} + \frac{\partial \vec{r}_i}{\partial t}\right\}$$

$$= \frac{\partial}{\partial q_j}\frac{d}{dt}(\vec{r}_i) = \frac{\partial \vec{v}_i}{\partial q_j}$$

So $\frac{d}{dt}$ & $\frac{\partial}{\partial q_j}$ commute.

Another useful result is that since

$$\vec{v}_i = \frac{d}{dt}\vec{r}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}$$

$$\Rightarrow \quad \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\delta_{kj} = \frac{\partial \vec{r}_i}{\partial q_j}$$

where δ_{kj} is the kronecker delta

$$\delta_{kj} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

Recall equation (F):

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left\{ \frac{d}{dt}\left(m \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right\}$$

we have shown we can swap $\frac{d}{dt}$ & $\frac{\partial}{\partial q_j}$ around

we have also shown that $\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \vec{v}_i}{\partial \dot{q}_j}$

Hence substituting we get

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left\{ \frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right\}$$

This equation can be further simplified:

$$\frac{\partial}{\partial \dot{q}_j}\left(m_i \vec{v}_i \cdot \vec{v}_i \right) = 2 m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}$$

$$\Rightarrow \quad m \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \left\{ \frac{1}{2} m_i v_i^2 \right\}$$

and similarly

$$m \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} = \frac{\partial}{\partial q_j} \left\{ \frac{1}{2} m_i v_i^2 \right\}$$

Hence we can combine these relations with equation (F) & the generalized force to re-write D'Alembert's Principle.

The result is

$$\sum_j \left\{ \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} \left\{ \sum_i \frac{1}{2} m_i v_i^2 \right\}\right) - \frac{\partial}{\partial q_j} \left\{ \sum_i \frac{1}{2} m_i v_i^2 \right\} - Q_j \right\} \delta q_j = 0$$

if we identify $\sum_i \frac{1}{2} m_i v_i^2 = T$ as the kinetic energy then

$$\sum_j \left\{ \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0$$

It can be shown that using the holonomic constraints we can find q_j that are independent. In this case all the terms in the $\{\ \} = 0$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

and there will be n equations.

Goldstein 1.4 cont.

Definition of the "Lagrangian"

Previously we derived that D'Alembert's Principle can be rearranged using generalized coordinates to give

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j}\right) = Q_j \qquad \text{——— (A)}$$

where T is the kinetic energy $T = \sum_i \frac{1}{2} m_i v_i^2$

We can yet further simplify this equation. The generalized force Q_j is defined by

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \qquad \text{——— (B)}$$

If $\vec{F}_i$ is the result of taking the gradient of a potential energy function V (e.g. classic example constant grav field $V = -mgy$ $-\frac{dV}{dy} = F = mg$) then we write

$$\vec{F}_i = -\nabla_i V \qquad \text{where } \nabla_i = \hat{i}\frac{\partial}{\partial x_i} + \hat{j}\frac{\partial}{\partial y_i} + \hat{k}\frac{\partial}{\partial z_i}$$

and V is the potential energy. Substituting in (B)

$$\Rightarrow \quad Q_j = \sum_i -\nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

However, since

$$\frac{\partial}{\partial q_j} = \sum_i \sum_{k=1}^{3} \frac{\partial r_i^k}{\partial q_j} \frac{\partial}{\partial r_i^k} = \sum_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \nabla_i$$

we find

$$Q_j = \frac{\partial V}{\partial q_j}$$

This can be substituted back into (A) to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$

This equation applies to any system with forces derivable from a potential including those for which V has an explicit time dependence. This is not the same as a conservative force where $V \equiv V(\vec{r})$, but such a potential would still apply here.
If we further assume that V is not a function of the generalized velocities $\dot{q}_j$ then we may write

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T-V)\right) - \frac{\partial}{\partial q_j}(T-V) = 0$$

We define $L = T - V$ to be the Lagrangian

and we have just shown that D'Alembert's Principle can be turned into the equation(s)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

This is called Lagrange's Equation(s) and since there are n q_j there will be n of them.

It is no coincidence that this equation is the same as the E-L equation we derived using the calculus of variations. Very soon we will connect the two using variational methods.

Note that the Lagrangian is not unique.

If we take any function $F(q_j, t)$ and define

$$L'(q_j, \dot{q}_j, t) = L(q_j, \dot{q}_j, t) + \frac{dF}{dt}$$

then L' also satisfies the E-L equation.

It is also important to note that we can still write down a Lagrangian $L = T - U$, even when a potential V does not exist, provided the generalized forces can be found via a function U where

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

This has important consequences in electrodynamics.

<u>Examples</u>

1) 1-d Harmonic oscillator using x as displacement

K.E. $T = \frac{1}{2} m v^2$
P.E. $V = \frac{1}{2} k x^2$

Lagrangian is defined as $L = T - V$
$= \frac{1}{2} m v^2 - \frac{1}{2} k x^2$

E.L. equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0$$

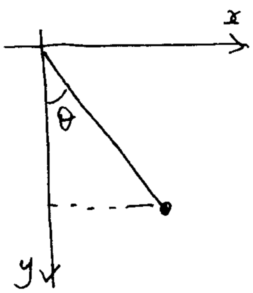
$$\Rightarrow \quad \frac{d}{dt}(m v) + k x = 0$$

$$\Rightarrow \quad m \frac{d^2 x}{dt^2} + k x = 0$$

(ie $F = -kx$ Hooke's Law)

So just by knowing the form of the kinetic & potential energy we can immediately write down the equations of motion!

2) Pendulum in Polar Coordinates



K.E. $\frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = T$

In polar coords $x = l \sin\theta$
$y = l \cos\theta$

Thus $\dot{x} = l \cos\theta \, \dot{\theta}$
$\dot{y} = -l \sin\theta \, \dot{\theta}$

Hence $T = \frac{1}{2} m (l^2 \cos^2\theta \, \dot{\theta}^2 + l^2 \sin^2\theta \, \dot{\theta}^2)$

$= \frac{1}{2} m l^2 \dot{\theta}^2$

Potential energy is $V = -mgy = -mgl\cos\theta$

Lagrangian is $L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$

Again "turn the crank" on the E-L equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2} m l^2 2\dot{\theta}\right) + mgl\sin\theta = 0$$

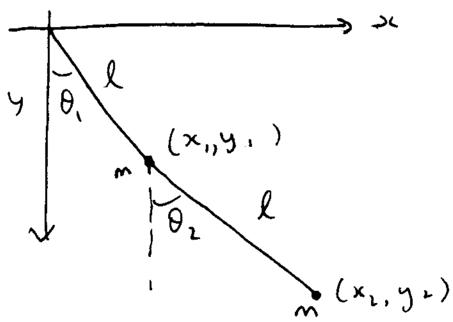
$$\Rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Notice how the tension in the string never appeared!

More complicated example:

<u>Double Pendulum</u>

Set both pendulum lengths & masses to be the same.



Resolve components as per usual:

$$x_1 = l \sin\theta_1 \quad \Rightarrow \quad \dot{x}_1 = l\cos\theta_1 \dot{\theta}_1$$
$$y_1 = l\cos\theta_1 \quad \Rightarrow \quad \dot{y}_1 = -l\sin\theta_1 \dot{\theta}_1$$

$$x_2 = l(\sin\theta_1 + \sin\theta_2) \Rightarrow \dot{x}_2 = l(\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2)$$
$$y_2 = l(\cos\theta_1 + \cos\theta_2) \Rightarrow \dot{y}_2 = -l(\sin\theta_1\dot{\theta}_1 + \sin\theta_2\dot{\theta}_2)$$

Writing out the kinetic energy:

$$T = \frac{1}{2}m(v_1^2 + v_2^2) = \frac{1}{2}ml^2\{\sin^2\theta_1\dot{\theta}_1^2 + \cos^2\theta_1\dot{\theta}_1^2\}$$
$$+ \frac{1}{2}ml^2\{\sin^2\theta_1\dot{\theta}_1^2 + 2\sin\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2 + \sin^2\theta_2\dot{\theta}_2^2$$
$$+ \cos^2\theta_1\dot{\theta}_1^2 + 2\cos\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 + \cos^2\theta_2\dot{\theta}_2^2\}$$
$$= \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\{\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\}$$
$$= \frac{1}{2}ml^2[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)]$$

Again taking potential energy to be zero at $y=0$

$$V = -mgy_1 - mgy_2 = -mgl\cos\theta_1 - mgl(\cos\theta_1 + \cos\theta_2)$$
$$= -mgl\,(2\cos\theta_1 + \cos\theta_2)$$

The Lagrangian is thus

$$L = \frac{1}{2}ml^2\left[2\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)\right] + mgl(2\cos\theta_1 + \cos\theta_2)$$

Next calculate $\frac{\partial L}{\partial\theta_1}, \frac{\partial L}{\partial\dot\theta_1}, \frac{\partial L}{\partial\theta_2}, \frac{\partial L}{\partial\dot\theta_2}, \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_1}\right), \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_2}\right)$

$$\frac{\partial L}{\partial\theta_1} = -ml^2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) - 2mgl\sin\theta_1$$

$$\frac{\partial L}{\partial\dot\theta_1} = 2ml^2\dot\theta_1 + ml^2\dot\theta_2\cos(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial\theta_2} = ml^2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) - mgl\sin\theta_2$$

$$\frac{\partial L}{\partial\dot\theta_2} = ml^2\dot\theta_2 + ml^2\dot\theta_1\cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_1}\right) = 2ml^2\ddot\theta_1 + ml^2\ddot\theta_2\cos(\theta_1-\theta_2) - ml^2\dot\theta_2(\dot\theta_1-\dot\theta_2)\sin(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_2}\right) = ml^2\ddot\theta_2 + ml^2\ddot\theta_1\cos(\theta_1-\theta_2) - ml^2\dot\theta_1(\dot\theta_1-\dot\theta_2)\sin(\theta_1-\theta_2)$$

Thus the equations of motion are

$$2\ddot\theta_1 + \ddot\theta_2\cos(\theta_1-\theta_2) + \dot\theta_2^2\sin(\theta_1-\theta_2) + \frac{2g}{l}\sin\theta_1 = 0$$

$$\ddot\theta_2 + \ddot\theta_1\cos(\theta_1-\theta_2) - \dot\theta_1^2\sin(\theta_1-\theta_2) + \frac{g}{l}\sin\theta_2 = 0$$

If we assume the small angle approximation for both θ_1 & θ_2, thereby keeping terms to first order in θ's then

$$\Rightarrow \left.\begin{aligned} 2\ddot{\theta}_1 + \ddot{\theta}_2 + \frac{2g}{l}\theta_1 &= 0 \\ \ddot{\theta}_2 + \ddot{\theta}_1 + \frac{g}{l}\theta_2 &= 0 \end{aligned}\right\} -(\mathrm{I})$$

A system with two degrees of freedom will have two solutions $\theta_1 = A_1 e^{i\omega t}$ $\theta_2 = A_2 e^{i\omega t}$. We have imposed the same frequency ω on both solutions though - what values of ω will allow this solution to work?

Substitute the trial solutions in —(I) to get

$$\left(\frac{2g}{l} - 2\omega^2\right) A_1 - \omega^2 A_2 = 0$$

$$-\omega^2 A_1 + \left(\frac{g}{l} - \omega^2\right) A_2 = 0$$

A_1 & A_2 are both non-zero (and unknown). Thus we need to solve for an ω that will simultaneously solve both equations.

Rearrange to get

$$(\omega^2)^2 - 4\frac{g}{l}\omega^2 + 2\frac{g^2}{l^2} = 0$$

Solve using quadratic formula:

$$\omega_\pm = \sqrt{2 \pm \sqrt{2}}\,\sqrt{\frac{g}{l}}$$

These $\omega_\pm$ define the <u>normal modes</u> of oscillation. Note that a double pendulum without the small angle approximation can be a <u>strongly chaotic system</u>.