There are two special cases that help simplify the E-L equations.

(1) $F \equiv F(x, y')$ so that F does not depend on y

In this case $\dfrac{\partial F}{\partial y} = 0$ and hence

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad \frac{\partial F}{\partial y'} = \text{constant}$$

This is what we observed for the shortest path problem. We now have a first order DE to solve which should obviously be much easier to handle.

(2) Suppose $F \equiv F(y, y')$ so that it has no explicit dependence on x, ie $\dfrac{\partial F}{\partial x} = 0$.

Starting from the E-L equation we have

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$\Rightarrow \quad y'\frac{\partial F}{\partial y} = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \qquad \text{——— (A)}$$

Since $\dfrac{d}{dx}\left(y'\dfrac{\partial F}{\partial y'}\right) = y''\dfrac{\partial F}{\partial y'} + y'\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right)$

$$\Rightarrow \quad y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) - y''\frac{\partial F}{\partial y'}$$

Substitute on LHS of (A) to get

$$y'\frac{\partial F}{\partial y} = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) - y''\frac{\partial F}{\partial y'} \qquad \text{——— (B)}$$

The next step is to write out $\frac{dF}{dx}$:

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{dy}{dx}\frac{\partial F}{\partial y} + \frac{d^2y}{dx^2}\frac{\partial F}{\partial y'}$$

$$= \frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'}$$

Since we have specified $\frac{\partial F}{\partial x} = 0$, we have

$$\frac{dF}{dx} = y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'}$$

Substituting in (B) yields

$$\frac{dF}{dx} = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)$$

$$\Rightarrow \quad \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$$

Integrate once to give

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}$$

So again we have derived a first order DE rather than second.

Examples:

Conside the shortest path problem of the 1^{st} lecture. We had to find the extremum of

$$I = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \Rightarrow \quad F = (1+y'^2)^{1/2}$$

Thus $\quad \dfrac{\partial F}{\partial y} = 0 \quad$ and $\quad \dfrac{\partial F}{\partial y'} = \dfrac{y'}{(1+y'^2)^{1/2}}$

hence the E-L equation gives

$$\frac{d}{dx}\left(\frac{y'}{(1+y'^2)^{1/2}}\right) = 0$$

$$\Rightarrow \qquad \frac{y'}{(1+y'^2)^{1/2}} = \text{constant} \qquad (\text{as before}).$$

Note that this result is particularly straightforward because F has no dependence on $y(x)$.

For the second example we shall consider the "Brachistochrone*" problem first posed (and solved) by Johann Bernoulli in 1696. The problem can be stated as:

Determine the curve through points A & B that will allow a ball rolling on the curve under gravity to complete the journey from A to B in the shortest possible time.

[* Brachistos = the shortest, chronos = time]

Potential energy of ball due to gravity is $-mgy$
Kinetic energy of ball is $\frac{1}{2}mv^2$

Since energy is conserved we must have

$$\frac{1}{2}mv^2 - mgy = \text{constant}$$

Scale y so that the constant is zero. (we need only add a constant in this case).

To work out the time taken we have to sum all the times to traverse each infinitesimal path length ds at speed v:

$$\text{Since} \quad \frac{ds}{dt} = v \quad \Rightarrow \quad dt = \frac{ds}{v}$$

$$\text{hence} \quad T = \int_0^T dt = \int_A^B \frac{ds}{v} \quad \text{between } A \,\&\, B$$

ds will again be given by decomposing into x & y components:

$$ds^2 = dx^2 + dy^2$$
$$\Rightarrow \quad ds = \sqrt{1 + y'^2}\, dx$$

For the velocity we substitute for v using

$$\frac{1}{2}v^2 = gy$$

$$\Rightarrow \quad v = \sqrt{2gy}$$

Thus $T = \int_A^B \frac{ds}{v} = \int_{x_A}^{x_B} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx = \int_0^{x_B} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$

The function $F(y, y')$ is given by $F(y,y') = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$

Since there is no explicit dependence on x, we can use our earlier result that

$$F - y'\frac{\partial F}{\partial y'} = \text{constant} = c \text{ ——— (d)}$$

Calculating $\frac{\partial F}{\partial y'}$ we get

$$\frac{\partial F}{\partial y'} = y'(1+y'^2)^{-1/2}(2gy)^{-1/2}$$

and subsituting into (d) gives

$$\frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - y'\frac{y'}{\sqrt{2gy}\sqrt{1+y'^2}} = c$$

$$\Rightarrow \quad \frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} = c$$

On squaring both sides & rearranging we get

$$y'^2 = \frac{k}{y} - 1 \Rightarrow y' = \sqrt{\frac{k}{y} - 1} \quad \text{where } k = \frac{1}{2c^2 g}$$

We can solve this equation by trying a parametric substitution:

Let $y = k\sin^2\theta = \frac{k}{2}(1-\cos 2\theta)$

$$\Rightarrow \frac{dy}{d\theta} = k\sin 2\theta$$

Then since $\frac{dy}{dx} = \frac{d\theta}{dx}\frac{dy}{d\theta} = \left(\frac{dx}{d\theta}\right)^{-1}\frac{dy}{d\theta}$

$$\Rightarrow \sqrt{\frac{k}{k\sin^2\theta} - 1} = \left(\frac{dx}{d\theta}\right)^{-1} . \; k\sin 2\theta$$

$$\frac{1}{\tan\theta} = \left(\frac{dx}{d\theta}\right)^{-1} . \; k\sin 2\theta$$

$$\Rightarrow \frac{dx}{d\theta} = k\sin 2\theta \; \tan\theta$$

$$= 2k \sin\theta \cancel{\cos\theta} \frac{\sin\theta}{\cancel{\cos\theta}}$$

$$= 2k\sin^2\theta = k(1-\cos 2\theta)$$

Integrate to get

$$x = \int^{\theta} k(1-\cos 2\theta)\, d\theta$$

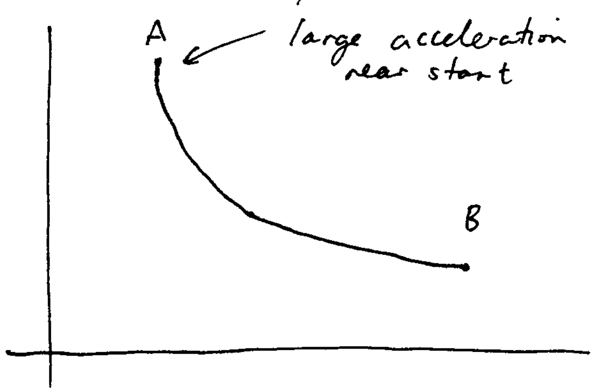
$$= k\left(\theta - \tfrac{1}{2}\sin 2\theta\right)$$

(constant set to zero by starting at $x=0$)

So we have a parametric solution:

$$\left.\begin{aligned} y &= \frac{k}{2}(1-\cos 2\theta) \\ x &= k\left(\theta - \frac{1}{2}\sin 2\theta\right) \end{aligned}\right\} \quad \text{where } \theta \in [0, \tfrac{\pi}{2}]$$

These are the parametric equations of an (inverted) cycloid.



A cycloid corresponds to the locus of a point on the circumference of a circle rotating along a fixed line.

Goldstein Sec 1.3

Constraints & Generalized Coordinates

Thus far we have developed some very powerful mathematical methodology. We now move onto creating creating the framework needed to describe complex dynamical systems.
While it is tempting to believe that all we really need to solve for any dynamical problem is

$$m\ddot{\vec{x}} = \vec{F}^e + \sum_{forces} \vec{F} \qquad \text{(ie a glorified } f = ma\text{)}$$

where $\vec{F}^e$ is an external force, and the sum over forces is over internal forces in the system, reality is frequently more complex.

Firstly, we usually need to deal with many body or many component systems, e.g. The solar system. In this case we need to solve a system of coupled equations

$$m_i \ddot{\vec{x}}_i = \vec{F}^e + \sum_j \vec{F}^{ij}$$

where the i-index denotes the equation for particle i, and the sum over j indicates a sum over all the other components/particles in the system. For the solar system example $\vec{F}^{ij}$ would be given by Newton's Law of Universal Gravitation.

Secondly, we may also have <u>constraints</u>. A simple example of a constraint is motion along a surface, another would be a rigid body whose components are constrained to be a certain distance apart. Yet another example is gas held within a container.

- Not all these constraints are "the same" some can be represented "simply" others cannot -

We classify constraints using the following nomenclature: If we can write the constraint in the form of an equation (or set of equations) as

$$f(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N, t) = 0$$

Where $\vec{r}_i$ is a vector describing the position of a component of the system, then the constraint is said to be holonomic.

holo = whole
nomos = management

Any other type of constraint is said to be non-holonomic. (Goldstein describes further classifications, but we are unlikely to need them.)

Example

For a rigid body the distance between parts of the body is fixed (e.g. ends on a dumbell). This can be expressed as a set of constraints

$$(\vec{r}_i - \vec{r}_j)^2 = c_{ij}^2$$

$$\Rightarrow \quad (\vec{r}_i - \vec{r}_j)^2 - c_{ij}^2 = 0$$

(i, j are just labels associated with the different parts.)

which is clearly a set of holonomic constraints.

We cannot write the constraint of being held in a container in this way (requires $\leq$ & $\geq$). Similarly, for a ball rolling on top of a sphere (to the point where it can fall off as well)

$$r^2 - a^2 \geq 0$$

this is non-holonomic because of the $\geq$ sign.

The key issue with holonomic constraints is that the coordinates used in the problem are no longer independent. The equation(s) of the constraint links them together. It also means that the equations of motion are not independent.

You may be thinking that constraints impose forces on the system - they do! However, a precise description of the force is not provided by the constraint. To find out what the force is you actually have to solve the system. However, by the very nature of the constraint you know how it affects the motion of the system - we'll give an example later.

To deal with the lack of independence of the coordinates we introduce the idea of generalized coordinates. Thus far we've thought (generally) in terms of cartesian coords, and if we have N parts/particles in a 3-d system then there are $3N$ degrees of freedom (3 x, y, z per part/particle).

If we have k holonomic constraints of the form

$$f(\vec{r}_1, \dots, \vec{r}_N, t) = 0$$

then the number of degrees of freedom is reduced to $3N-k$. Hence we can view the system is actually having $3N-k$ <u>generalized coordinates</u>.

We can also write the $\vec{r}_i$ implicitly (ie parametrically) in terms of the $3N-k$ generalized coordinates which are traditionally denoted q_j

ie

$$\vec{r}_1 = \vec{r}_1(q_1, q_2, \dots, q_{3N-k}, t)$$

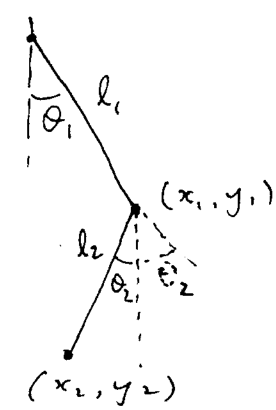
$$\vdots \qquad \vdots$$

$$\vec{r}_N = \vec{r}_N(q_1, q_2, \dots, q_{3N-k}, t)$$

Note that unlike 3-d cartesian coordinates, generalized coordinates do not come in groups of 3.

<u>Example</u>

Consider the double pendulum problem:



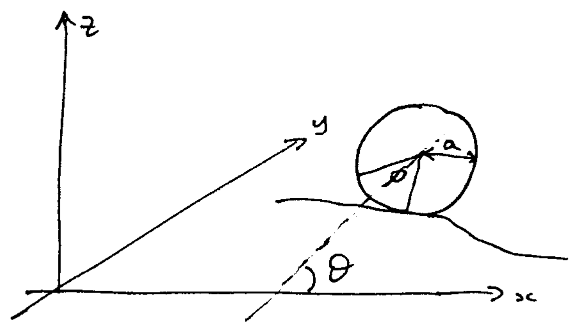
Pendulum 1 has position (x_1, y_1)
" 2 " " (x_2, y_2)

However they are not independent & can be described in terms of the angles (θ_1, θ_2):

$$(x_1, y_1) = (l_1 \sin\theta_1, \ l_1 \cos\theta_1)$$

$$(x_2, y_2) = (l_1 \sin\theta_1 + l_2 \sin\theta_2, \ l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

Non-holonomic constraints are often described in terms of relationships between <u>velocities</u> rather than coordinates. A classic example is that of a vertical disk rolling on a plane.

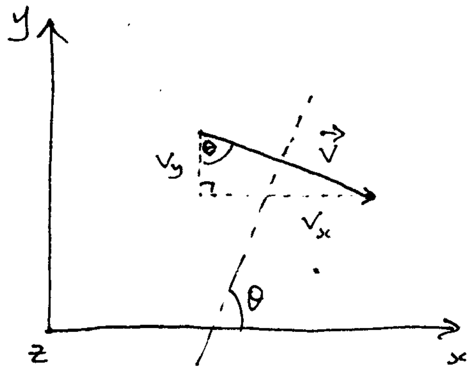


Rate of angular rotation of disk is $\dot{\phi}$

Axis of disk makes an angle θ with x-axis.

Radius of disk = a

From above:



Speed of disk = radius × angular speed

$$v = a\dot{\phi} = a\frac{d\phi}{dt}$$

Using the above diagram to resolve the velocity components:

$$V_x = \dot{x} = v \sin\theta$$
$$V_y = \dot{y} = -v\cos\theta$$

Since $dt = \frac{a}{v}\, d\phi$ we get

$$dx - a\sin\theta\, d\phi = 0$$
$$dy + a\cos\theta\, d\phi = 0$$

So the constraint that the disk rolls vertically along the plane is expressed between the differentials of the system.

Holonomic constraints are always amenable to a "formal solution" in the sense that the method to solve the system is well formulated. For non-holonomic constraints the situation is more complicated. A special method called "Lagrange multipliers" allows us to use the differential constraint along with the differential equation of motion. We will return to this soon.

Ultimately, to make our derivations as simple as possible we want to formulate the system in such a way that any constraint forces disappear. This problem was first considered by James Bernoulli, but was developed by D'Alembert.