Goldstein Chap 11

Introduction to Chaos

We have spent a great deal of time studying systems with "integrable" equations (ie we can find a solution in closed form). In practice most physical systems of interest are not integrable. While approximation methods, e.g. small angles, help, such solutions are only of limited validity.

Perturbation theory is typically applied when we can separate a problem into two pieces. The first piece has a simple & integrable solution while the second part of the system is much smaller in magnitude & thus a small perturbation. What happens when the perturbative part is no longer small? Perturbation theory breaks down & we must consider the full system without any approximations.

In such cases, numerical solutions to the equations of motion are necessary. However, this fact alone does not mean a system will behave in a "chaotic" fashion. What do we mean by chaotic though? For a system described by a Hamiltonian then a system is "chaotic" if phase space trajectories separated by an infinitesimal amount initially tend to widely varying phase-space trajectories over time. This "varying" of the trajectories is something that we can discuss in a quantitative fashion (see next lecture).

It is very important to note that the behaviour of chaotic systems is deterministic – we are numerically integrating a specific set of equations of motion. While the resulting behaviour may look "random" in some ways, it most certainly isn't. This has lead some people to consider chaos is lying between "randomness" and stable solutions. Of course for classical systems true randomness is a misnomer – classical systems are by definition deterministic.

Poincaré Sections

Most technical discussion of chaos focuses on the phase space behaviour of systems. Since this is a $2n$ dimensional space we need a tool to help understand motion in this space more easily.

Consider a Hamiltonian associated with two uncoupled oscillators, so that we have 4 phase space coordinates. We can write such a system as

$$H = \frac{p_1^2}{2m_1} + \frac{1}{2} m_1 \omega_1^2 q_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2} m_2 \omega_2^2 q_2^2$$

In this case the total energy is

$$E_T = E_1 + E_2$$

where both E_1 & E_2 are conserved quantities of the motion. Hence although 4 coordinates are used to describe the system in practice it will be constrained to a $4-2=2$ dimensional surface.

If we scale the coordinates by

$$p_i \Rightarrow \frac{p_i}{(2m_i)^{1/2}}$$

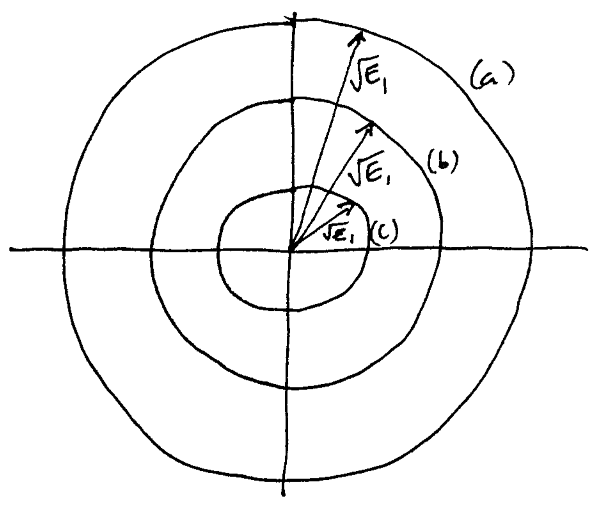
$$q_i \Rightarrow q_i\left(\tfrac{1}{2}m_i\omega_i^2\right)^{1/2}$$

then we can scale out the mass & frequency dependence to give

$$q_i^2 + p_i^2 = E_i$$

which is the equation of a circle of radius $\sqrt{E_i}$.

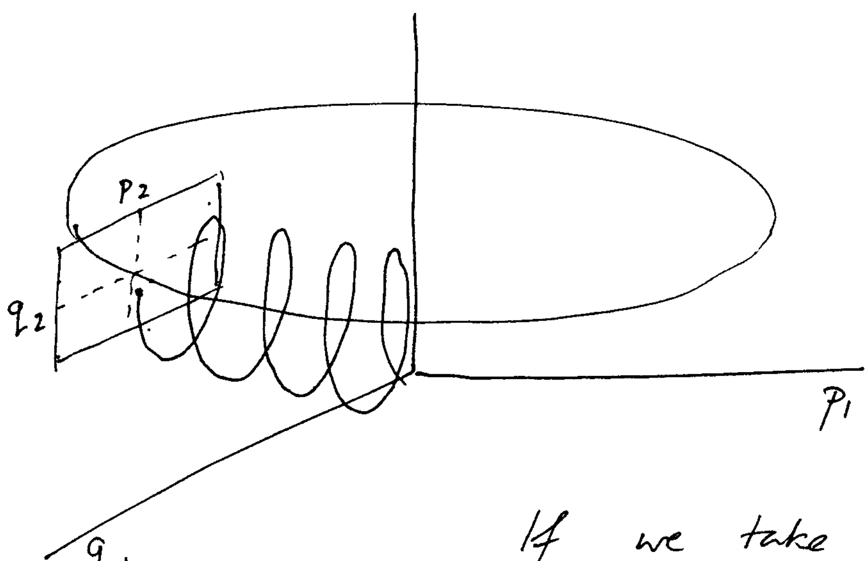
If E_T is kept fixed we may still vary the relative contributions of E_1 & E_2. In turn this will change the relative sizes of circles in the plots of q_i & p_i, e.g. consider a plot of q_1, p_1



For circle (a) $q_1^2 + p_1^2$ is large $\Rightarrow$ E_1 dominates over E_2 ($E_1 > E_2$)
For (b) $q_1^2 + p_1^2$ is intermediate ($E_1 \sim E_2$)
For circle (c) $q_1^2 + p_1^2$ is small ($E_1 < E_2$)

For convenience consider $\omega_2 \gg \omega_1$. Oscillator $\underline{1}$ behaves as we have discussed above. Oscillator 2 must behave in a similar fashion when plotted in the q_2, p_2 plane, with the size of the circle determined by the relative energy contributions.

This overall motion can be drawn in 3d. In fact as stated the systems evolution in phase space is such that it follows a 2-d surface. In general though, if energy is conserved the system will evolve on a $2n-1$ dimensional surface called the energy hypersurface.



The overall motion of the system is a spiralling motion around the larger orbit associated with oscillator 1.

If we take a cross section through the phase space (e.g. the q_2, p_2 patch shown) then the system will pass through each cross section at different points. In this example the system will loop around in the q_1, p_1 plane & pass through the q_2, p_2 cross section again & again.

As the system passes through the cross section it may build up a plot like the q_1, p_1 plot we considered or it may just be a bunch of points lying on the q_2, p_2 track if we ignored q_1 & p_1 (ie if we projected over all q_1, p_1).

These sections clearly help us to understand the overall motion in phase space & are called Poincaré sections. For our example the structure of the section is very simple (points on a circle) but for chaotic systems Poincaré sections can be very complex. Note the term Poincaré map is sometimes used to describe the collection of points in a Poincaré section.

Henon-Heiles Hamiltonian

The origins of this particular Hamiltonian relate to a desire to understand stellar motion in the potential of the galaxy. In these systems both total energy & total angular momentum are conserved. Henon & Heiles were looking for more constants of the motion & found some that existed, but only under certain circumstances.

The Hamiltonian they considered is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}k(x^2+y^2) + \lambda\left(x^2y - \frac{1}{3}y^3\right) \quad \text{———} \; (A)$$

λ would normally be assumed to be small and thus make it a perturbation on the overall motion. However, Henon & Heiles allowed it to be large relative to the other terms. No analytic solutions exist for the equations of motion.

Set $m=1$, $\lambda=1$, $k=1$ to simplify things as much as possible. Then

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + x^2y - \frac{1}{3}y^3$$

the equations of motion are the

$$\ddot{x} = -x - 2xy$$
$$\ddot{y} = -y - x^2 + y^2$$

which are clearly coupled & non-linear.

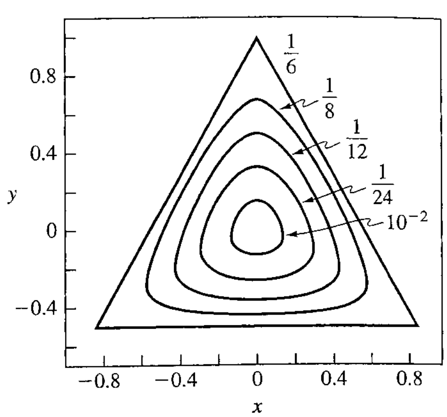
Let's look at the behavior of the system using Poincaré sections. ~~Fi~~ What about the behavior in the x, y projection?

Going back to (A) we can express the potential in polar coords to get

$$V(r,\theta) = \frac{1}{2}r^2 + \frac{1}{3}r^3 \sin 3\theta = r^2\left(\frac{1}{2} + \frac{1}{3} r \sin 3\theta\right)$$

analysis of this cubic equation shows that for a fixed V r is maximized when $\sin 3\theta = -1$, and r is minimized when $\sin 3\theta = 1$.

If we keep V fixed & look at the equipotential in terms of x & y we can then put together an overall picture of this potential.



In the limit $E = \frac{1}{6}$ the curves become an equilateral triangle

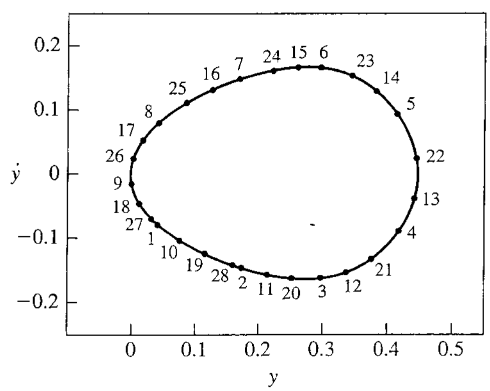
For $E < \frac{1}{6}$ the curves get progressively closer to circles.

For $E > \frac{1}{6}$ the curves are no longer open & diverge to infinity.

Let's now put a Poincaré section across $x=0$, so that we consider the $y, \dot{y}$ cross section. Taking $E = 1/12$ & choosing starting values of $\dot{y}_1 = 0.08$, $y = 0.02$, we find the initial $\dot{x}_1$ is

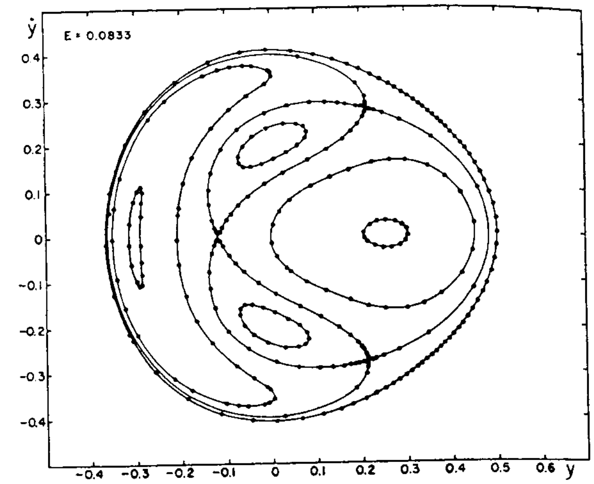
$$\dot{x}_1 = \left(2E - \dot{y}_1^2 - y_1^2 + \frac{2}{3}y_1^2\right)^{1/2}$$

A numerical solution technique is then employed to calculate when the system passes through the Poincaré section again. This is repeated producing a sequence of points $(\dot{y}_2, y_2)$, $(\dot{y}_3, y_3)$, $(\dot{y}_4, y_4)$



Notice how the points all appear to lie on a oval type shape.

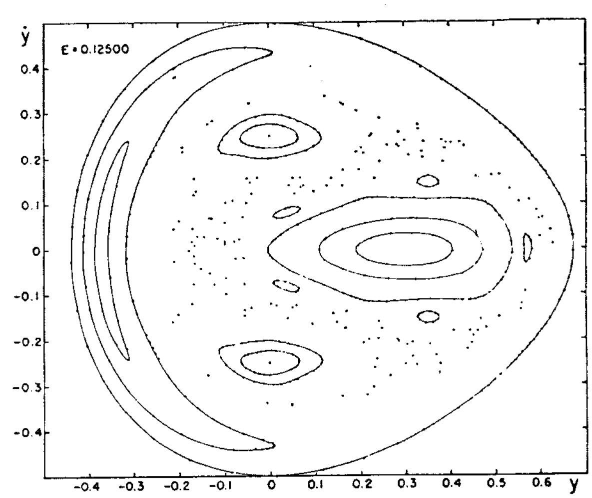
Henon & Heiles then examined different starting $\dot{y}$ & y and found the following behavior in the Poincaré Section.



(a)

There are four regions with "oval shaped" orbits & $\dot{y}=0$ is a line of symmetry.

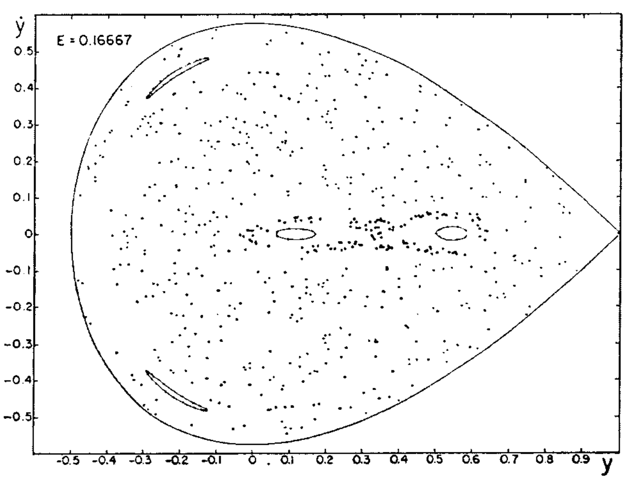
What happens as we change E?
Increasing to $E = 1/8$ & repeating the Poincaré section calculation gives



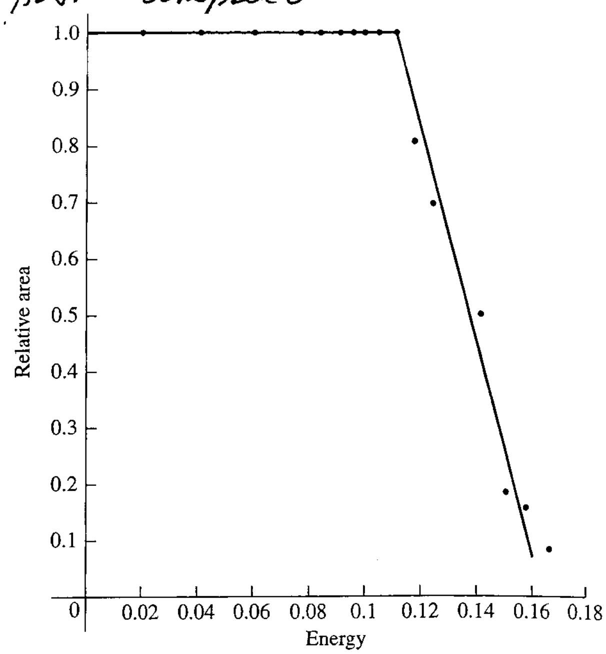
Notice that while the oval regions still exist there are numerous "dots" appearing & regions that do not look "well behaved." In these areas as the system passes through the Poincaré section it appears to do so almost randomly. It can jump from one part of the Poincaré section to another.

Increasing to $E = 1/6$ sees a further reduction in size of the oval regions & an increase in "random" areas.

(b)



We can plot up the relative area of the "well behaved" part compared to the total area of the Poincaré section

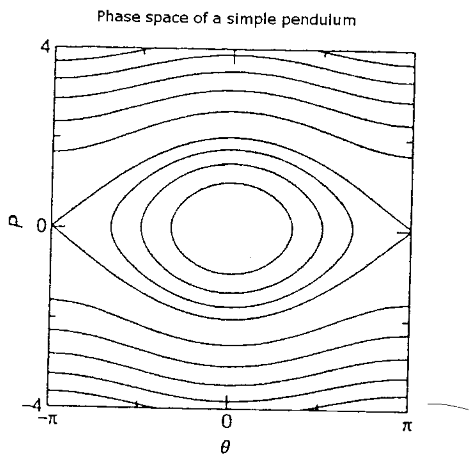


This shows a critical E $(\sim \frac{1}{6})$ at which point there are no well behaved regions!

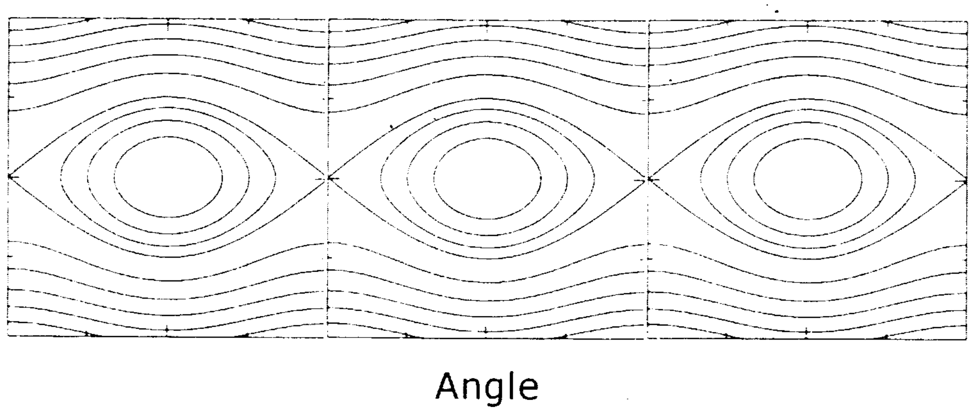
Phase Space Structures & Chaos

The origins of chaos are deeply rooted in the possible trajectories of systems in phase space. Hence let us consider a few examples to see what kind of behaviour is possible.

Firstly, consider a pendulum in 1-d, the phase space diagram is as follows



In fact the full phase space extends over many multiples of 2π:



Within each of the oval regions (separated by 2π) looping around the oval corresponds to a libration of the pendulum. The very centre of the oval at $p=0$, $\theta=0$ (or for that matter $\theta = 2\pi n$) is completely stable in the sense that motion does not proceed rapidly away from it.
Increasing the amplitude of the libration serves to increase the size of the oval until p reaches a critical value. If p is large enough to complete a loop without reversing then the angular momentum must always be +ve or -ve. Hence the system proceeds along the lines above or below the oval regions.

What about the "X" points at $-3\pi, -\pi, \pi, 3\pi$ etc?

Exactly at each of these points the pendulum is perfectly inverted & hence exactly stable. Hence these points are also "fixed points". However the points are completely unstable to any small perturbations since that will either make the pendulum loop around repeatedly (ie we push it 'above' the ovals) or it will librate on a very long swing (we move it inside the oval regions). Hence this is called an unstable fixed point.

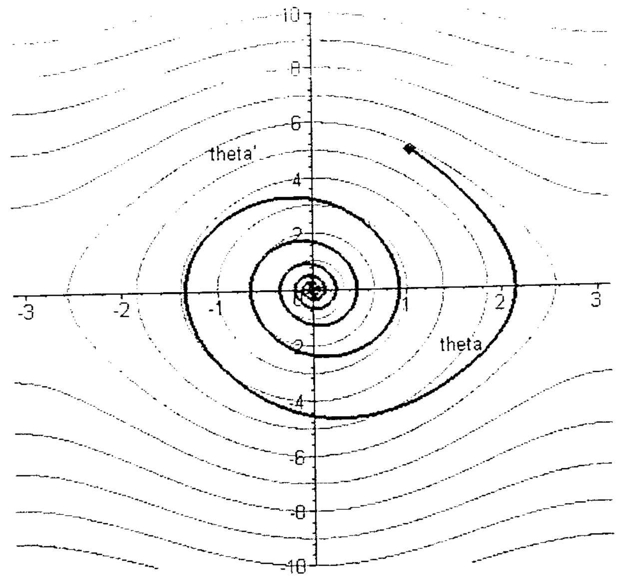
The X's clearly separate regions of different physical behaviour & trajectories in phase space & are called <u>separatrices</u>

If you look back at the Henon-Heiles Poincaré sections you can see separatrices there.

If we add drag to the librating pendulum, which can be represented by (normalized units)

$$\dot{\theta} = p$$
$$\dot{p} = -\sin\theta - 0.1\dot{\theta}$$

gives an inspiralling solution toward the fixed point.



Because the system evolves towards a specific part of the phase space we call the fixed point a "fixed point attractor."

Attractors do not need to be points either. Systems can evolve toward a particular repeated trajectory or "limit cycle."
For example the van der Pol equation

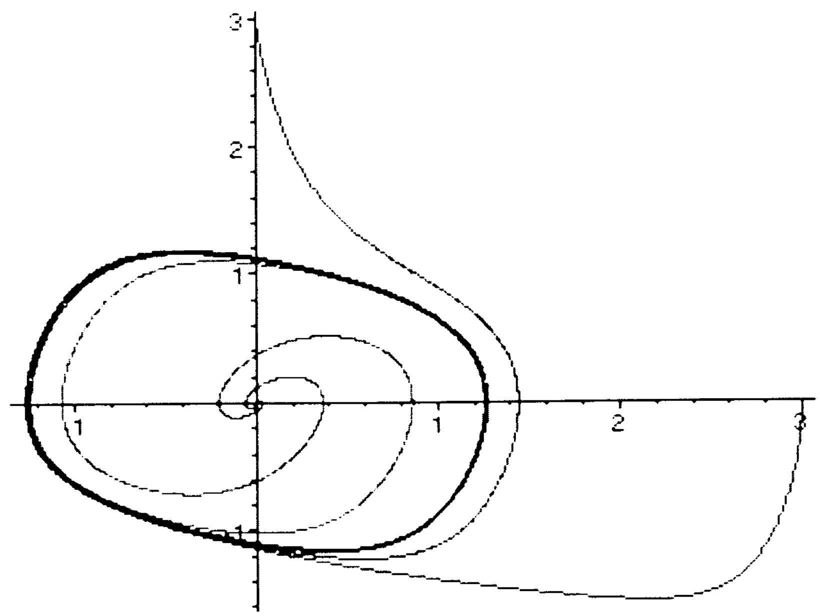
$$m\ddot{x} - \epsilon(1-x^2)\dot{x} + m\omega_0^2 x = F\cos\omega_D t$$

Where

ω_0 = frequency of driving force
ϵ = damping parameter
ω_0 = natural resonant frequency of oscillator

In short, if $x^2 > 1$ damping is +ve
$\Rightarrow$ system spirals inward toward "limit cycle"

if $x^2 < 1$ damping is -ve
$\Rightarrow$ system spirals outward toward "limit cycle"



Lastly, in higher dimensional systems attractors can have a very dispersed & disjointed structure. Such attractors are called "strange attractors" & are intimately connected with chaos. These attractors can be "fractal" in nature. A classic example is the Lorenz Attractor.

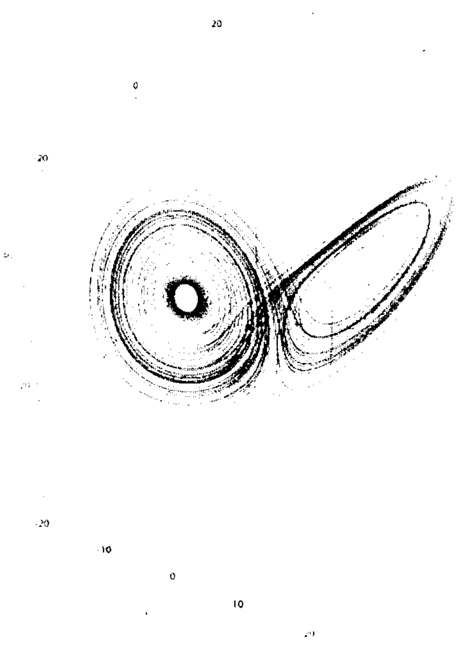
Lorenz Attractor (First derived in Lorenz (1963))

Arose from trying to model convection in the atmosphere. The simplified equations governing the system are

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(\rho - z) - y\\ \dot{z} &= xy - \beta z\end{aligned}$$

σ = Prandtl number (ratio of viscous to thermal diffusion coefficients)
ρ = Rayleigh number (associated with buoyancy)
β = geometric factor

$\sigma, \rho, \beta > 0$ & typically $\sigma = 10$, $\beta = 8/3$ but ρ is allowed to vary depending on a given model.
Starting from two very similar positions trajectories eventually take very different paths. However, both paths follow the strange attractor.



Liapunov exponents

We started the previous lecture by stating that chaos arises when small separation in phase space become widely separated in a given period of time. This essentially encapsulates the "Butterfly effect" concept - a tiny change leads to very different long term behaviour.

To quantify this idea, suppose two initial configurations of a system in phase space are separated by $|\delta \vec{s}_0|$ (distance being measured in $2n$ dimensions). Over time the distance between these two points is $|\delta \vec{s}(t)|$ (by defn). The Liapunov exponent then classifies the behaviour of this distance with time according to

$$|\delta \vec{s}(t)| \approx e^{\lambda t} |\delta \vec{s}_0|$$

Thus for $\lambda > 0$ the separation grows until the limits of the phase space are reached
$\Rightarrow$ chaos

For $\lambda = 0$, this corresponds to a limiting case at the edge of chaos.

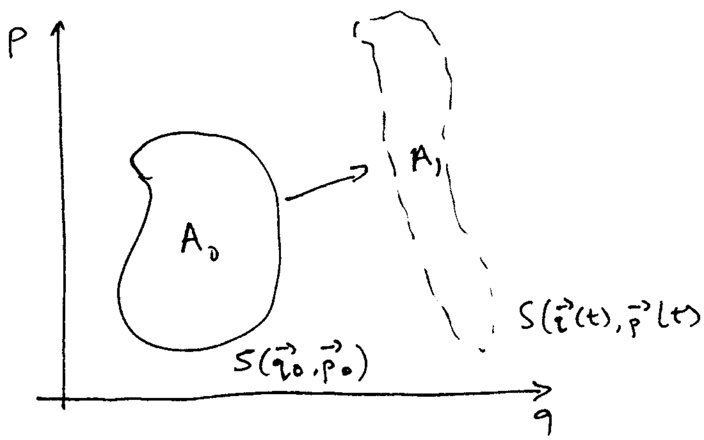
For $\lambda < 0$, trajectories converge indicating the presence of a (non-strange) attractor.

Roughly speaking the Liapunov exponent is a measure of the relative distortion of phase space over time (ie as a system evolves).

Although we have not shown it, the integral over phase space coordinates

$$J = \iint_{S(\vec{p},\vec{q})} d\vec{p}\, d\vec{q}$$

is conserved over time. This implies if we choose a boundary of a region in phase space the "area" within that region will stay constant even though the position of the boundary changes with $\vec{q}(t)$, $\vec{p}(t)$ evolution.



"Area" $A_0 = A_1$

For chaotic systems this region will become extremely stretched & distorted.