Goldstein Chap 9.

Canonical Transformations

In deriving Hamilton's equations of motion from the action integral we noted that while the proof required δq_i vanish at the end points, the same was not required for δp_i.
However, if we do require that δp_i vanish at the end points we find an interesting consequence. Suppose we add a function $\frac{dF}{dt}(q_i, p_i, t)$ to the Lagrangian, then under variations we find

$$\delta I = \delta\int_{t_1}^{t_2} \left(L + \frac{dF}{dt}(q_i, p_i, t) \right) dt$$

$$= \delta\int_{t_1}^{t_2} L \, dt + \delta\int_{t_1}^{t_2} \frac{dF}{dt} \, dt$$

$$= \delta\int_{t_1}^{t_2} L \, dt + \delta\left[F(q_i, p_i, t) \right]_{t_1}^{t_2}$$

$$= \delta\int_{t_1}^{t_2} L \, dt + \left[\frac{\partial F}{\partial q_i}\delta q_i + \frac{\partial F}{\partial p_i}\delta p_i \right]_{t_1}^{t_2}$$

then if $\delta q_i(t_1) = \delta q_i(t_2) = 0$
& $\delta p_i(t_1) = \delta p_i(t_2) = 0$

we get

$$\delta\int_{t_1}^{t_2} \frac{dF}{dt} \, dt = \left[\frac{\partial F}{\partial q_i} \times 0 + \frac{\partial F}{\partial p_i} \times 0 - \frac{\partial F}{\partial q_i} \times 0 - \frac{\partial F}{\partial p_i} \times 0 \right]$$

$$= 0$$

Hence $$L + \frac{dF}{dt} = \dot{q}_i p_i - H(q_i, p_i, t) + \frac{dF}{dt}$$

is also an acceptable Lagrangian &

$$H'(q_i, p_i, t) = H(q, p, t) - \frac{dF}{dt}$$

is also an acceptable Hamiltonian.

This result, while seemingly innocuous has some profound implications.

Consider two coordinate systems describing the same dynamical system, e.g. sometimes we use x, y & sometimes r, θ in 2d. If we consider a system with a central force law, so that the potential is $V(r)$, then while the x, y formulation includes terms in x & y, the r, θ formulation will not include terms in θ (only $\dot{\theta}$). Hence the r, θ formulation has 1 cyclic variable.

In Hamiltonian mechanics cyclic variables are particularly informative. Consider the case where the Hamiltonian is constant in time & all the q_i are cyclic. Then $\dot{p}_i = 0 \Rightarrow p_i = \alpha_i$ and hence the Hamiltonian can be written

$$H(p_i, q_i, t) = H(\alpha_i)$$

The equations of motion are then $\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial \alpha_i} = \omega_i$

where the ω_i are functions of the α_i only &

Therefore constant in time. Thus

$$\dot{q}_i = \omega_i \quad \Rightarrow \quad q_i = \omega_i t + \beta_i$$

where the β_i are constants of integration.

So if we can find coordinate transformations that bring the Hamiltonian into a form with cyclic variables we can find solutions for systems rapidly. Let's consider the process of transforming variables in more detail.

Suppose we have a set of coordinates q_i, p_i with Hamiltonian $H(q_i, p_i, t)$ & another set of coordinates Q_i, P_i with Hamiltonian $K(Q_i, P_i, t)$. Suppose we can write the Q_i, P_i:

$$Q_i = Q_i(q_j, p_j, t)$$
$$P_i = P_i(q_j, p_j, t)$$

such a transformation is called a "point transformation of phase space."

Since Q_i, P_i are equally valid coordinates, Hamilton's Principle is

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q_i, P_i, t)) \, dt = 0$$

while for the q_i, p_i

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q_i, p_i, t)) \, dt = 0$$

it is tempting, but incorrect to the conclude

$$\cancel{p_i \dot{q}_i - H = P_i \dot{Q}_i - k}$$

this is not correct for two reasons. Firstly since the RHS of Hamilton's Principle is zero we can include an arbitrary scaling factor, λ ie

$$\lambda (p_i \dot{q}_i - H) = P_i \dot{Q}_i - K$$

Secondly, we must consider the arbitrariness of the definition of the Lagrangian that we showed earlier: L is only specified up to an arbitrary function $\frac{dF}{dt}(q_i, p_i, t)$ {or equivalently an $\frac{dF'}{dt}(Q_i, P_i, t)$}. Thus

$$\lambda (p_i \dot{q}_i - H(p_i, q_i, t)) = P_i \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF}{dt}$$

In fact F can be a function of any of the phase space coordinates & must have at least continuous second derivatives in those coordinates.

Note that λ defines a "scale transformation." If we let

$$Q_i = \mu q_i \qquad (\mu, \nu \text{ constant})$$
$$P_i = \nu p_i$$

then from Hamilton's equations of motion

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \Rightarrow \mu \dot{q}_i = \frac{\partial K}{\partial (\nu p_i)} \Rightarrow \dot{q}_i = \frac{\partial}{\partial p_i}\left(\frac{1}{\nu\mu} K\right)$$

and since $\dot{q}_i = \frac{\partial H}{\partial p_i}$ we have $H = \frac{1}{\nu\mu} K$

ie $K = \mu\nu H$

thus we have

$$\mu\nu(p_i \dot{q}_i - H) = \dot{Q}_i \hat{P}_i - K$$

and hence we identify $\lambda = \mu\nu$. In general such transformations are uninteresting & we will be concerned with coordinate transformation obeying

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt} \qquad \text{——— (A)}$$

which are called "canonical transformations."

Since the variation of q_i, p_i, Q_i, P_i are all zero at the end points F can be an explicit function of any of them

When F is a function of one old variable and one of the "new" it actually specifies a "bridge" between the two sets of variables & is called a "generating function" of the transformation.

Consider $F \equiv F_1(q_i, Q_i, t)$, from (A) we get

$$p_i \dot{q}_i - H(q_j, p_j, t) = P_i \dot{Q}_i - K(P_j, Q_j, t) + \frac{dF_1}{dt}$$

expanding $\frac{dF_1}{dt}(q_i, Q_i, t)$:

$$p_i \dot{q}_i - H(q_j, p_j, t) = P_i \dot{Q}_i - K(Q_j, P_j, t) + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t}$$

rearranging

$$\dot{q}_i \left\{ p_i - \frac{\partial F_1}{\partial q_i} \right\} - \dot{Q}_i \left\{ P_i + \frac{\partial F}{\partial Q_i} \right\} = H(q_j, p_j, t) - K(Q_j, P_j, t) + \frac{\partial F_1}{\partial t}$$

q_i & Q_i vary independently, and the RHS is not a function of either. Thus the coefficients on the LHS ahead of the $\dot{q}_i$ & $\dot{Q}_i$ must vanish.

$$\Rightarrow \quad p_i = \frac{\partial F_1}{\partial q_i}(q_j, Q_j, t) \qquad \text{——} (B)$$

$$\Rightarrow \quad P_i = -\frac{\partial F_1}{\partial Q_i}(q_j, Q_j, t) \qquad \text{——} (C)$$

and consequently

$$K(Q_j, P_j, t) = H(q_j, p_j, t) + \frac{\partial F_1}{\partial t} \qquad \text{——} (\delta)$$

Thus give an $F_1(q_j, Q_j, t)$ then (B) gives p_i as a function of the q_j, Q_j, t. Assuming this result is invertible this can then be used to give the Q_i's as a function of the q_j's, p_j's.
This result can then be used to substitute into (C) to give the P_i's as a function of the q_j's & p_j's.

Then (D) gives the new Hamiltonian K as a function of H & $\frac{\partial F_1}{\partial t}$.

Note that we must first express H in terms of Q & P using the inversion of the $Q_i(q_j, p_j, t)$ & $P_i(q_j, p_j, t)$ just found & lastly we do the same for $\frac{\partial F_1}{\partial t}$.

Thus starting from a single function $F_1(q_j, Q_j, t)$ we can derive the equations for the transformed variables.

More on Canonical Transformations

What about the other possibilities for F?

$$F_2 \equiv F_2(q_i, P_i, t)$$
$$F_3 \equiv F_3(p_i, Q_i, t)$$
$$F_4 \equiv F_4(p_i, P_i, t)$$

we can relate these functions back to $F_1(q_i, Q_i, t)$ by using the same Legendre transformation we used in defining the Hamiltonian.

Recall the L.T. allow used to encapsulate the function $Y(X)$ into an equation $c(x)$ in its derivative $x(X) = \frac{dY}{dX}$ using

$$c(x) = xX - Y(X)$$

or equivalently $\quad -c(x) = Y(X) - x.X$

Then if we define

$$F_2(q_j, P_j, t) = F(q_j, Q_j, t) + P_i Q_i$$

then $\quad P_i = -\frac{\partial F}{\partial Q_i}$. So we let

$$F \equiv F_2(q_j, P_j, t) - P_i Q_i$$

we then find that

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K - \dot{P}_i Q_i - P_i \dot{Q}_i + \frac{d}{dt} F_2(q_j, P_j, t)$$

$$= -\dot{P}_i Q_i - K + \frac{d}{dt} F_2(q_j, P_j, t)$$

$$p_i \dot{q}_i - H = -\dot{P}_i Q_i - K + \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i + \frac{\partial F_2}{\partial t}$$

Setting coefficients of $\dot{q}_i$ & $\dot{P}_i$ to zero (as before) we get

$$p_i = \frac{\partial F_2}{\partial q_i}$$

$$Q_i = \frac{\partial F_2}{\partial P_i}$$

$$\& \qquad K = H + \frac{\partial F_2}{\partial t}$$

by the same process of Legendre transformation we find

$$F \equiv F_3(p_j, Q_j, t) + q_i p_i$$
$$F \equiv F_4(p_j, P_j, t) + q_i p_i - Q_i P_i$$

For the F_3 type generator we find

$$q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i}$$

& for the F_4 type generator

$$q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i}$$

Lets consider a trivial example of a canonical transf. first & then look at the harmonic oscillator in more detail.

Trivial example

Let's consider the $F_2(q_j, P_j, t)$ generator where

$$F_2(q_j, P_j, t) = q_i P_i$$

Then immediately from the earlier calculation

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j$$
$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j$$
$$\& \quad K = H$$

Thus F_2 is said to generate an identity transformation. There is a similar example for F_3, but for F_1 & F_4

$$\text{let } F_1 = q_i Q_i \Rightarrow Q_j = p_j \qquad P_j = -q_j$$
$$F_4 = p_i Q_i \Rightarrow Q_j = p_j \qquad P_j = -q_j$$

Harmonic Oscillator

We write the Hamiltonian as

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \qquad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Notice that if we could find a canonical transformation which leads to a form

$$\left.\begin{aligned} p &= f(P)\cos Q \\ q &= \frac{f(P)}{m\omega}\sin Q \end{aligned}\right\} - (I)$$

Then the Hamiltonian becomes

$$K = H = \frac{f^2(P)}{2m}\left(\cos^2 Q + \sin^2 Q\right) = \frac{f^2(P)}{2m}$$

and then Q is clearly cyclic. We of course still need to find a generating function that can produce this form.

Consider

$$F_1 = \frac{m\omega}{2} q^2 \cot Q$$

applying the canonical transformation equation for the F_1 type generator we find

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q$$

$$P = \frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}$$

Solving for q & p in terms of P&Q:

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q \qquad p = \sqrt{2Pm\omega}\cos Q$$

The equating back to (I) we identify

$$f(P) = \sqrt{2m\omega P}$$

Since $\frac{\partial F_1}{\partial t} = 0$, we see immediately that K is simply

$$K = \frac{2m\omega P}{2m} = \omega P$$

Hence the Hamiltonian is now cyclic in Q. Further since $E = H = K$ then we immediately have

$$P = \frac{E}{\omega}$$

and applying Hamilton's equations of motion

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega$$

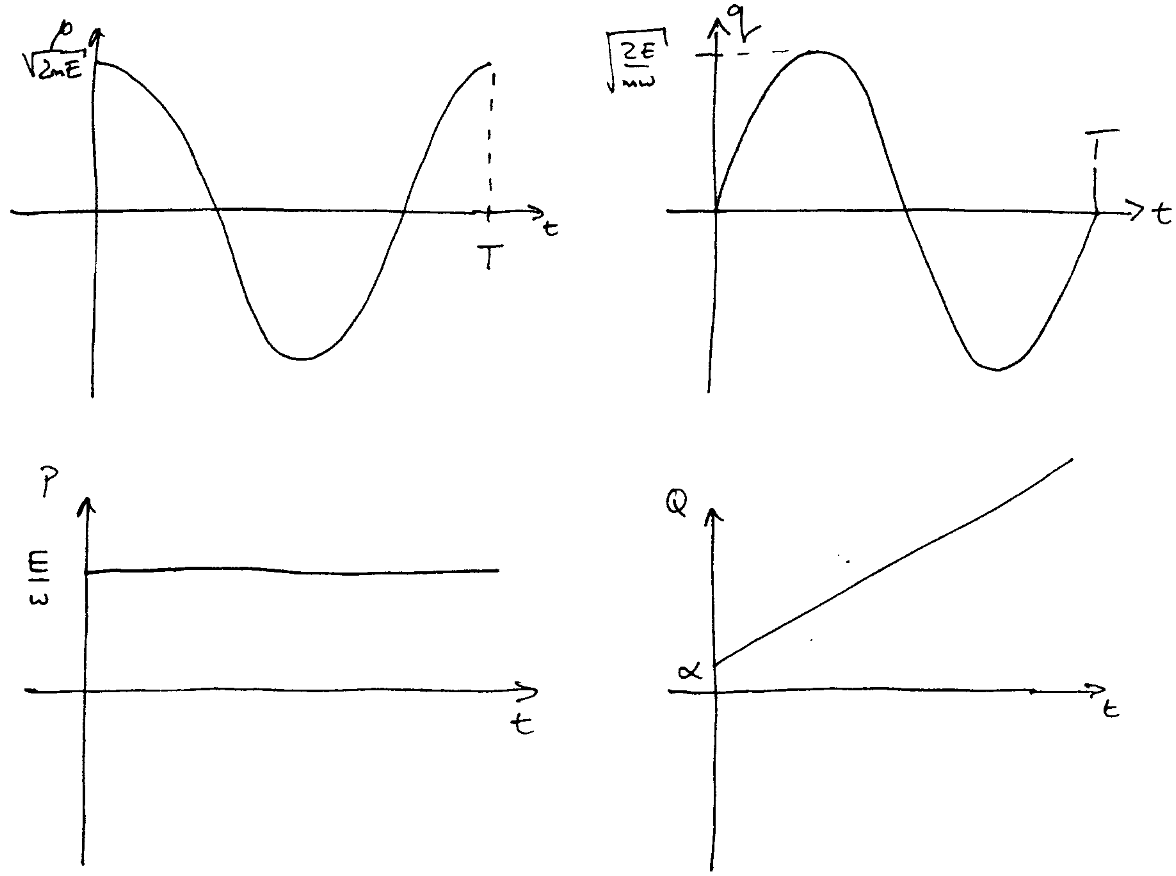
$$\Rightarrow \quad Q = \omega t + \alpha$$ where α is a constant of integration

Thus our solutions for p & q are

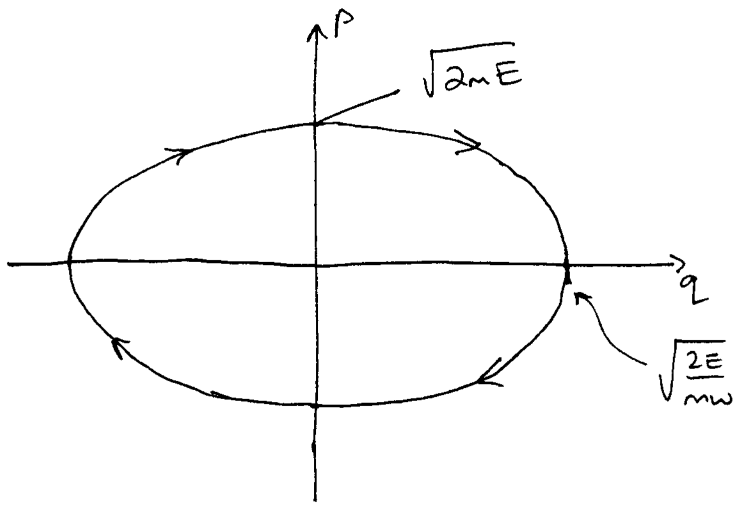
$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha)$$

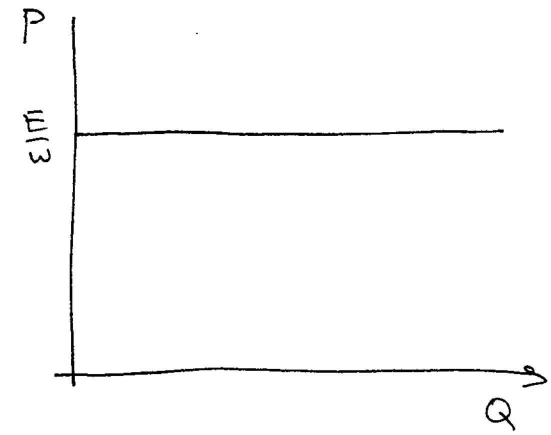
$$p = \sqrt{2mE} \cos(\omega t + \alpha)$$

The time behaviour of the two sets of coordinates is clearly very different:



We can also draw the phase spaces for the two systems.





Incidentally the origin of the term phase space should be clear from those plots. In the transformed coordinates the variable Q effectively gives the phase of the oscillation at any one time.

This example is admittedly somewhat trivial, but nonetheless it shows how the idea works in practice. See Chapter 11 of Goldstein for more details on applications.

Lastly in this section let us look a the relationship between Poisson Brackets & canonical transformations.

Under the definition of the P.B. it is straightforward to show that

$$\{q_j, q_k\} = 0 = \{p_j, p_k\}$$

and

$$\{q_j, p_k\} = \frac{\partial q_j}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \cancel{\frac{\partial q_j}{\partial p_i}\frac{\partial p_k}{\partial q_i}} = \delta_{ji}\delta_{ki} = \delta_{jk}$$

These three relationships are called the fundamental Poisson Brackets.

What does $\{Q_j, P_k\}$ equal for the results of a canonical transformation?

Lets consider only transformation where

$$Q_j = Q_j(q_i, p_i) \qquad \& \text{ inverting } \quad q_j = q_j(Q_i, P_i)$$
$$P_j = P_j(q_i, p_i) \qquad\qquad\qquad\qquad p_j = p_j(Q_i, P_i)$$

The the time derivative of Q_j is

$$\dot{Q}_j = \frac{\partial Q_j}{\partial q_i}\dot{q}_i + \frac{\partial Q_j}{\partial p_i}\dot{p}_i = \frac{\partial Q_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial Q_j}{\partial p_i}\frac{\partial H}{\partial q_i} \quad \text{—(II)}$$

from Hamilton's equation of motion.

expanding $\frac{\partial H}{\partial P_j}$ we find

$$\frac{\partial H}{\partial P_j} = \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial P_j} + \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial P_j} \quad \text{———(III)}$$

since for the transformed Q_j, P_j to be new & equivalent canonical variables we must have

$$\dot{Q}_j = \frac{\partial H}{\partial P_j} \qquad \text{(from Hamilton's eqns)}$$

we can equate (II) & (III) to get

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial p_j}{\partial P_i} \qquad \& \qquad \frac{\partial Q_i}{\partial p_j} = -\frac{\partial q_j}{\partial P_i}$$

a similar procedure for P_j gives

$$\frac{\partial P_i}{\partial q_j} = -\frac{\partial p_j}{\partial Q_i} \quad \& \quad \frac{\partial P_i}{\partial p_j} = \frac{\partial q_j}{\partial Q_i}$$

Hence we can now evaluate $\{Q_j, P_k\}$:

$$\{Q_j, P_k\} = \frac{\partial Q_j}{\partial q_i}\frac{\partial P_k}{\partial p_i} - \frac{\partial Q_j}{\partial p_i}\frac{\partial P_k}{\partial q_i} = \frac{\partial Q_j}{\partial q_i}\frac{\partial q_i}{\partial Q_k} + \frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial Q_k}$$

$$= \frac{\partial Q_j}{\partial Q_k} = \delta_{jk}$$

We also find by the same process of substitution that

$$\{Q_j, Q_k\} = \{P_j, P_k\} = 0$$

Hence the fundamental Poisson Brackets are invariant under canonical transformations.

Hence you can test whether a transformation is canonical or not by taking the P.B. of the transformed variables.

We can also show that under a canonical transformation

$$\{u, v\}_{\text{in } q,p \text{ coords}} = \{u, v\}_{\text{in } Q,P \text{ coords}}$$

ie P.B.'s are invariant under canonical transformations.