Then
$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial}{\partial \dot{x}^\alpha}\left\{\frac{1}{2} m_0 \eta_{\mu\nu} u^\mu u^\nu\right\}$$
$$= \frac{1}{2} m_0 \left\{\eta_{\mu\nu}\, \delta^\mu_\alpha\, u^\nu + \eta_{\mu\nu}\, \delta^\nu_\alpha\, u^\mu\right\}$$
$$= \frac{1}{2} m_0 \left\{\eta_{\alpha\nu} u^\nu + \eta_{\mu\alpha} u^\mu\right\}$$
$$= m_0 u_\alpha$$

hence
$$\frac{d}{d\tau}(m_0 u_\alpha) = 0$$

ie 4-momentum is conserved. Hence $L$ defines an acceptable free-particle Lagrangian.

Including potentials is also quite difficult. For example, if we wanted to include gravity then we would need to use General Relativity. However, electromagnetic potentials from Maxwell's equations can be included in a straightforward way:

Let
$$L = \frac{1}{2} m_0 u_\mu u^\mu + q u^\mu A_\mu$$

where $q$ = charge on particle
$A_\mu$ = 4-vector of electromagnetic potential

From Maxwell's Equations
$$\vec{E} = \nabla\phi - \frac{\partial \vec{A}}{dt} \qquad \phi = \text{electric potential}$$
$$\vec{B} = \nabla \times \vec{A} \qquad \vec{A} = \text{magnetic potential}$$

Then $A_\mu = \left(\frac{\phi}{c}, \vec{A}\right)$

Applying the E-L equations, we first calculate derivatives,

$$\frac{\partial L}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left(q u^\mu A_\mu\right)$$

$$\frac{\partial L}{\partial \dot{x}^\nu} = m_0 u_\nu + q A_\nu$$

then substitute into the E-L equations:

$$\frac{\partial}{\partial x^\nu}\left(q u^\mu A_\mu\right) - \frac{d}{d\tau}\left(m_0 u_\nu\right) - q\frac{dA_\nu}{d\tau} = 0$$

$$\Rightarrow \qquad \frac{dp_\nu}{d\tau} = -q\frac{dA_\nu}{d\tau} + \frac{\partial}{\partial x^\nu}\left(q u^\mu A_\mu\right)$$

This ends our brief sojourn into relativistic Lagrangians!

Goldstein Chap 8

## Introduction to Hamiltonian Mechanics

Hamiltonian mechanics (~1833) was developed by the Irish mathematician William Hamilton roughly 50 years after the development of Lagrangian mechanics. The principle advantage of Hamiltonian mechanics is a conceptual one, it provides a great deal of insight into the dynamics of systems. It also provides the foundation for the development of Hamiltonians in quantum theory & classical perturbation theory.

However, as a tool for finding explicit solutions to equations of motion, it is not actually any better than Lagrangian mechanics. However, the conceptual advantages outweigh this issue.

As a motivation for Hamiltonian mechanics, since the E-L equations of motion are

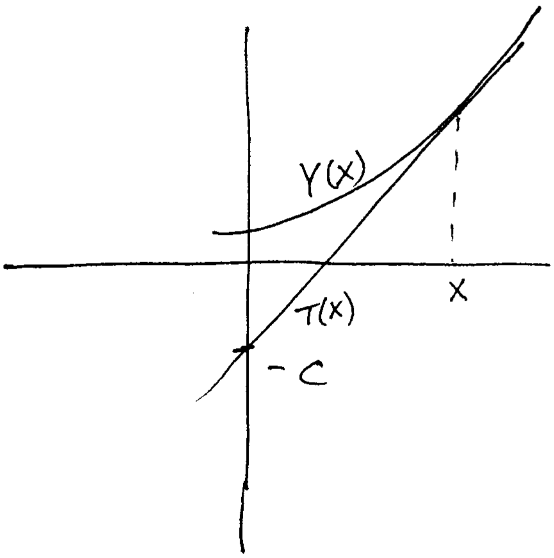
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

we could equally well write

$$\frac{d}{dt}(p_i) = \frac{\partial L}{\partial q_i} \qquad \text{where } p_i \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{q}_i}$$

this suggests that we could equally describe the evolution of the system in terms of $p_i$'s & $q_i$'s rather than $q_i, \dot{q}_i$. Lets look at how we can form a function of $p_i$'s & $q_i$'s that gives the behavior we want.

For simplicity consider a simple function $Y(X)$ first. Define the variable conjugate to $X$ by $x = \frac{dY}{dX}$ (in analogy with $p_i = \frac{\partial L}{\partial \dot{q}_i}$).

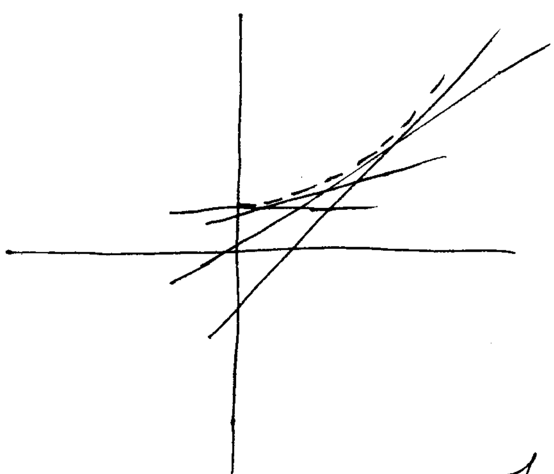



At any point on $Y(X)$ we can define a tangent

$$T(X) = \frac{dY}{dX} \cdot X - c = xX - c$$

where $-c$ is the intercept.

As $X$ changes so does the slope $x$ & the intercept $-c$ required to produce the tangent at $Y(X)$. You can think of an infinite set of these tangent lines actually forming $Y(X)$:

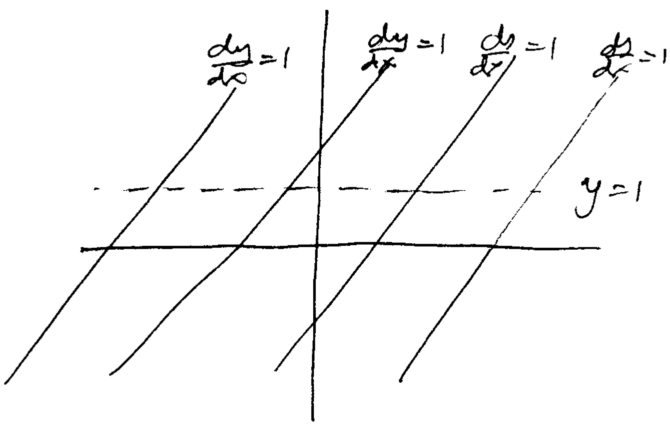


Notice how the intercept is as important as the slope for determining $Y(X)$. Without the information in the intercept we cannot fully specify $Y(X)$. Thus we can't just rewrite $Y(X)$ as a new function $Y(x(X))$, we need the intercept information.

If this isn't clear, think about how we define $y(x)$ - for every $x$-value we have a map to $y(x)$.

If we construct a table of values of $y\left(\frac{dy}{dx}\right)$ and $\frac{dy}{dx}$ what does this mean geometrically?

Suppose we have $y=1$ & $\frac{dy}{dx}=1$, then




we have a family of $\frac{dy}{dx}=1$ slopes, all of which obey $y=1$ (at some point) and $\frac{dy}{dx}=1$.

Thus the need for the intercept information should be clear.

Is the intercept in fact all we need? Yes!

Since at $Y(X)$:

$$Y(X) = T(x) = x.X - c$$

$$\Rightarrow \quad c(x) = xX - Y(X)$$

Note you can confirm it is only a function of $x$ as follows:

$$dc(x) = Xdx - X\,dx - dY$$

but $\quad x\,dX = dY$

$$\Rightarrow \quad dc = X\,dx$$

If this isn't clear, remember $c$ is the position of the intercept as a function of the slope - it has to be the map on to $y(x)$!

There is one slight complication - if there are two values of $Y(X)$ that map to the same intercept but have different slopes (ie $x$ values) then the 1-to-1 map breaks down.

The situation doesn't occur as long as $Y[X]$ is a convex function on an interval $[a, b]$ ie

$$Y''(X) \geqslant 0 \quad \forall \quad X \in [a, b]$$

Hence given our definition of

$$c(x) = xX - Y(X)$$

we need to write this out & then eliminate the factors of $X$ using $x = \frac{dY}{dX}$ to arrive at a function of $x$ only.

The process of transforming $Y(X)$ into the function $c(x)$ is called a <u>Legendre Transformation</u>. [Strictly speaking I have used a slightly different terminology for the intercept ($-c$ versus $c$)]

Let's apply this idea to a Lagrangian, $L(q_i, \dot{q}_i, t)$ where the $q_i$ are $n$ degrees of freedom. Canonical momenta are defined as usual by
$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Define the Hamiltonian by

$$H(q_i,p_i,t) = \sum_{i=1}^{n} \dot{q}_i p_i - L(q_i,\dot{q}_i,t) \qquad \text{— (A)}$$

[note: Goldstein uses the summation convention!]

This is exactly analogous to the transformation we derived but in more variables.

What equations does this transformation imply for the Hamiltonian?

Beginning from $L(q_i,\dot{q}_i,t)$ we have that

$$dL = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

(from now on use the summation convention)

$$\therefore \quad dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

$$= \frac{\partial L}{\partial q_i} dq_i + p_i\, d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

since $p_i = \frac{\partial L}{\partial \dot{q}_i}$

From (A) we get

$$dH = d\dot{q}_i\, p_i + \dot{q}_i\, dp_i - dL(q_i,\dot{q}_i,t)$$

$$= \cancel{d\dot{q}_i\, p_i} + \dot{q}_i\, dp_i - \cancel{p_i\, d\dot{q}_i} - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt$$

$$= \dot{q}_i\, dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt$$

From the E-L equations we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{d}{dt}(p_i) = \frac{\partial L}{\partial q_i}$$ and subbing in (B) gives

$$dH = \dot{q}_i \, dp_i - \dot{p}_i \, dq_i - \frac{\partial L}{\partial t} dt$$

But since $H \equiv H(q_i, p_i, t)$ we also have

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

Equating coefficients then gives

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \end{aligned}\right\}$$ Hamilton's equations of motion.

Thus we have turned the $n$ second order equations in the E-L equation into $2n$ first order equations.

We might view $\dot{q}_i = \frac{\partial H}{\partial p_i}$ is not providing much new information because it looks like an inverse of $p_i = \frac{\partial L}{\partial \dot{q}_i}$. However, in the Hamiltonian formalism both of these equations are equally important & valid.

How do we use this formalism?

We proceed in 5 steps:

1) Write down $L = T - V$ as usual in terms of $q_i, \dot{q}_i, t$

2) Calculate canonical momenta from $p_i = \frac{\partial L}{\partial \dot{q}_i}$

3) Form the Hamiltonian using $H = p_i \dot{q}_i - L$

4) Invert the $p_i = \frac{\partial L}{\partial \dot{q}_i}$ relationship to give $\dot{q}_i$ as function of $q_i, p_i, t$

5) Eliminate the $\dot{q}_i$ from $H$ using the results of step 4.

Example:

Step (1) Consider 1-d simple harmonic oscillator. Unstretched length of spring $= l$, mass $m$ is attached to spring.

$$T = \frac{1}{2} m \dot{x}^2 \qquad V = \frac{k}{2}(x - l)^2$$

$$\Rightarrow \qquad L = \frac{1}{2} m \dot{x}^2 - \frac{k}{2}(x - l)^2$$

Step 2) Canonical momenta:

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

Step 3) Form the Hamiltonian

$$H = \dot{x}p - \frac{1}{2}m\dot{x}^2 + \frac{k}{2}(x-\ell)^2$$

Step 4) Since $p = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$

Step 5) $H = \frac{p^2}{m} - \frac{1}{2}m\frac{p^2}{m^2} + \frac{k}{2}(x-\ell)^2$

after substituting $\dot{x} = \frac{p}{m}$

$$\Rightarrow \quad H = \frac{1}{2m}p^2 + \frac{k}{2}(x-\ell)^2$$

Lets now look at Hamilton's equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \Rightarrow \quad \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \Rightarrow \quad \dot{p} = -k(x-\ell)$$

As I mentioned earlier, the first of the equations doesn't really feel like it gives us anything new. If we substitute for $p$ in the second equation

$$\Rightarrow \quad m\ddot{x} = -k(x-\ell)$$

ie $\quad m\ddot{x} + k(x-\ell) = 0$

Which is what we would have derived from the E-L equations – as hoped!

## More on Hamiltonians

For the simple harmonic oscillator we showed that

$$H(q,p) = \frac{1}{2m}p^2 + \frac{k}{2}(q-\ell)^2 \qquad (x \equiv q)$$

Since $\frac{k}{2}(q-\ell)^2 = V(q)$

and $\frac{1}{2m}p^2 = \frac{1}{2}m\dot{x}^2 = T(\dot{x}) \equiv T(p)$

then $H(q,p) = T(p) + V(q) = E$

where $E$ is the total energy.

We can see why $H$ is conserved in certain situations by looking at the Lagrangian. If we have a system where $\frac{\partial L}{\partial t} = 0$, then

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} \qquad \text{(summation convention)}$$

$$= \frac{d}{dt}(p_i)\frac{dq_i}{dt} + p_i\frac{d\dot{q}_i}{dt} \qquad \text{from E-L equations}$$

$$\Rightarrow \quad \frac{d}{dt}(p_i\dot{q}_i - L) = \frac{d}{dt}H = 0$$

$\therefore \quad H =$ constant

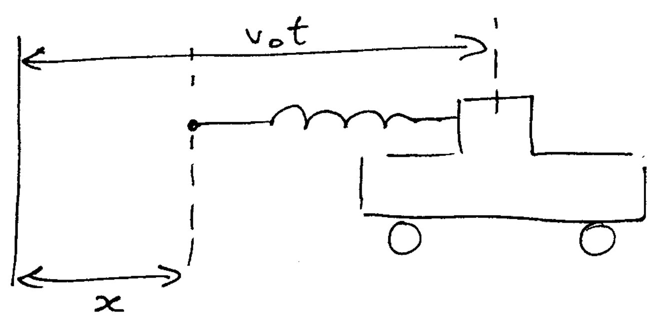
Thus $\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0$ & by Hamilton's eqn's $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0$

Many texts start by defining the Hamiltonian as the total energy $E = T + V$, but it doesn't have to be. That said, if you have a Lagrangian with $\frac{\partial L}{\partial t} = 0$ then you can find a Hamiltonian equal to the total energy.

Nonetheless, in many cases we can define $H = E$, then if $\frac{\partial L}{\partial t} = 0$ we know this is also a conserved quantity.

When might the Hamiltonian <u>not</u> be the total $E$? It depends on how the generalized coordinates are defined.

<u>Example</u>
Suppose we attach a spring to a massless moving cart with position $v_0 t$ ($v_0$ = constant). The end of the spring is attached to a mass $m$, which has position $x$:




Suppose spring has unstretched length $l_0 = 0$.

The Lagrangian is

$$L = \frac{1}{2} m \dot{x}^2 - \frac{k}{2}(x - v_0 t)^2$$

E-L equations give

$$m\ddot{x} = -k(x - v_0 t)$$

Notice that $\frac{\partial L}{\partial t} \neq 0$ in this case.

Suppose we define a generalized coordinate
$$x' = x - v_0 t$$
then in this case the equation of motion is
$$m\ddot{x}' = -kx'$$

$x'$ is the displacement of the mass relative to the cart. Hence an observer moving with the cart sees the mass undergo S.H.M.

The Hamiltonian for this system can be quickly shown to be (using the $x$ coordinate) not $x'$

$$H = \frac{1}{2m} p^2 + \frac{k}{2}(x - v_0 t)^2$$

where $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ & I have used $x$ instead of $q$ – it doesn't matter.

Notice that $H$ is not constant – but since $\frac{\partial L}{\partial t} \neq 0$ this should not be a surprise. However, $H$ is the total energy.

$H$ ~~increases~~ is not constant because the cart being driven at speed $v_0$ is actively doing work against force of the spring during an oscillation.

Can show that $\frac{dH}{dt} = \frac{\partial H}{\partial t} = \frac{k}{2} v_0^2 t - \frac{k v_0 x}{2}$

If we rework the system using $x'$ as the generalized coordinate, then $\dot{x} = \dot{x}' + v_0$ & the Lagrangian is then

$$L = \frac{1}{2} m\dot{x}'^2 + m v_0 \dot{x}' + \frac{1}{2} m v_0^2 - \frac{k}{2} x'^2$$

Now $p' = \frac{\partial L}{\partial \dot{x}'} = m(\dot{x}' + v_0)$ and we get

$$H = \frac{1}{2m} (p' - m v_0)^2 + \frac{k}{2} x'^2 - \frac{m v_0^2}{2}$$

We can ignore the $m v_0^2/2$ term since it is a constant & doesn't affect the equation of motion. However, this new Hamiltonian $\underline{\underline{\text{is}}}$ constant since $\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} = \frac{dH}{dt} = 0$, even though we've just shown the total energy is not conserved! This Hamiltonian corresponds to the total energy of the particle moving relative to the cart.

Notice, importantly, that despite these two Hamiltonians having different forms they still imply the same equations of motion.

We have shown that if properly defined Hamiltonians correspond to the total energy of a system, and under the situation $dH/dt = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0$ this is conserved. What about cyclic coordinates & conserved momenta?

Recall a cyclic coordinate is one where $\frac{\partial L}{\partial q_i} = 0$. This implies a conserved momentum $p_i$ since by the E-L equation $\dot{p}_i = \frac{\partial L}{\partial q_i} = 0$.

Since $H = \dot{q}_j p_j - L(\dot{q}_j, q_j, t)$ assuming $L$ is not a function of $q_i$ but $j \neq i$

$$\Rightarrow \quad \frac{\partial H}{\partial q_i} = \frac{\partial \dot{q}_j}{\partial q_i} p_j - \dot{q}_j \frac{\partial p_j}{\partial q_i} - \frac{\partial L}{\partial q_i}$$

$$= 0$$

So $H$ is also independent of $q_i$. From Hamilton's equations we then have

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0$$

as expected. So we expect exactly the same type of conservation behavior of the canonical momenta.

## Hamiltonian from a Variational Principle

Before we derive the Hamiltonian by this method note that $H(p_i, q_i, t)$ describes the evolution of the system in what is called phase space - $2n$ dimensional space if there are $n$ degrees of freedom (since $q_i$'s, $p_i$'s come in pairs). This is a doubling of the equivalent space for Lagrangians, which we called configuration space. However, Hamilton's equations of motion constrain the evolution of $q_i$ & $p_i$

ensuring that we have not introduced additional degrees of freedom.

Phase space is extremely important in discussing dynamics of systems. It is used extensively in astrophysical dynamics (for example).

Recall Hamilton's Principle is that the action integral

$$I = \int_{t_1}^{t_2} L \, dt \quad \text{is extremized, ie that}$$

$$\delta I = \delta \int_{t_1}^{t_2} L \, dt = 0$$

Since $\quad H = p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$

$$\Rightarrow \quad L = p_i \dot{q}_i - H(q_j, p_j, t)$$

Thus $\quad \delta I = \delta \int_{t_1}^{t_2} \left( p_i \dot{q}_i - H(q_j, p_j, t) \right) dt$

$$= \int_{t_1}^{t_2} \left\{ \delta p_i \, \dot{q}_i + p_i \, \delta \dot{q}_i - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right\} dt$$

$$= \int_{t_1}^{t_2} \left\{ \delta p_i \left( \dot{q}_i - \frac{\partial H}{\partial p_i} \right) + \left[ p_i \, \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i \right] \right\} dt$$

Since $q$ is equivalent to the spatial coordinate which has zero variation at the end points

$$\int_{t_1}^{t_2} \left( p_i \frac{d}{dt}(\delta q_i) - \frac{\partial H}{\partial q_i} \delta q_i \right) dt = \int_{t_1}^{t_2} \delta q_i \left\{ -\dot{p}_i - \frac{\partial H}{\partial q_i} \right\} dt + \cancel{\left[ p_i \, \delta q_i \right]_{t_1}^{t_2}}$$

Thus

$$\delta I = \int_{t_1}^{t_2} \left[ \left\{ \delta p_i \left\{ \dot{q}_i - \frac{\partial H}{\partial p_i} \right\} + \delta q_i \left\{ -\dot{p}_i - \frac{\partial H}{\partial q_i} \right\} \right] dt = 0$$

Hence we must have that

$$\dot{q}_i - \frac{\partial H}{\partial p_i} = 0 \qquad \Rightarrow \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$-\dot{p}_i - \frac{\partial H}{\partial q_i} = 0 \qquad \Rightarrow \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

ie Hamilton's equations!

Notice that we didn't need to specify that the variations in $\delta p_i$ vanished at the end points.

## Poisson Brackets

Let us quickly define a very useful operator. Suppose we are given a function of phase space coordinates $f \equiv f(q_j, p_j, t)$. The

$$\frac{df}{dt} = \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial t}$$

using Hamilton's equations of motion.

Let us define

$$\{f, H\} \stackrel{\text{def}}{=} \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}$$ as the Poisson bracket of $f$

then $$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \quad \text{—— (A)}$$

Notice that by definition $\{f, H\} = -\{H, f\}$

We can also write (A) in an "operator" form:

$$\frac{d}{dt} f = \left(\frac{\partial}{\partial t} - \{H, \cdot\}\right) f$$

and if we take $f = H$ then

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} - \{H, H\} = \frac{\partial H}{\partial t}$$ as expected.

This operator is important for defining so called "canonical transformations" which is what we will look at next.