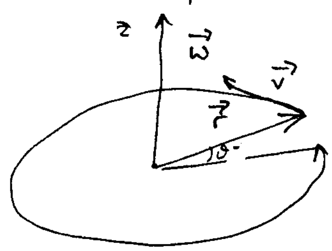
Euler's Equations

Here is a quick demonstration of $\vec{v} = \vec{\omega} \times \vec{r}$ using cylindrical polar coordinates.



Taking $\vec{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix}$

Then $\vec{v} = \dfrac{d\vec{r}}{dt} = \begin{pmatrix} \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \\ \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} \\ 0 \end{pmatrix} = \dot{r}\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + r\dot{\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$

but by definition $\hat{r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$ and $\hat{\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$

(resolve components for yourself).

$\therefore \quad \vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$ & if $\dot{r} = 0 \Rightarrow \vec{v} = r\dot{\theta}\,\hat{\theta}$

By definition $\omega = \dfrac{d\theta}{dt} = \dot{\theta}$

Let us take $\vec{\omega} = \omega\,\hat{z}$, Then the cross-product with $\vec{r}$ gives

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ r\cos\theta & r\sin\theta & 0 \end{vmatrix} = -\omega r\sin\theta\,\hat{i} + \omega r\cos\theta\,\hat{j}$$

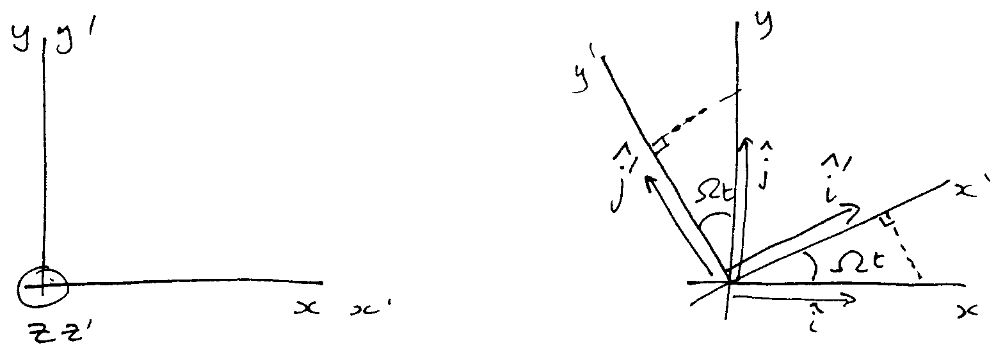
$$\equiv \omega r\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = \omega r\,\hat{\theta} = \vec{v}$$

$\therefore \quad \vec{v} = \vec{\omega} \times \vec{r}$

Relating time derivatives from a non-rotating coordinate system or "frame" to a rotating one is slightly non-trivial. Let the non-rotating frame by (x, y, z) while the rotating frame has coordinates (x', y', z'). Assume the rotating frame has as an angular velocity of Ω and that at $t = 0$ $\quad x' = x, \; y' = y$.

From above:

$t = 0$



time t

Suppose we have unit vectors $\hat{i}', \hat{j}', \hat{k}'$ which describe the unit vectors for the (x', y', z') but we write them in terms of $\hat{i}, \hat{j}$.

Resolving components of $\hat{i}, \hat{j}$ gives

$$\left.\begin{aligned} \hat{i}'(t) &= \cos(\Omega t)\,\hat{i} + \sin(\Omega t)\,\hat{j} \\ \hat{j}'(t) &= -\sin(\Omega t)\,\hat{i} + \cos(\Omega t)\,\hat{j} \end{aligned}\right\} \quad (A)$$

Working out the time derivatives of these vectors:

$$\left.\begin{aligned} \frac{d}{dt}\hat{i}'(t) &= -\Omega\sin(\Omega t)\,\hat{i} + \Omega\cos(\Omega t)\,\hat{j} \\ \frac{d}{dt}\hat{j}'(t) &= -\Omega\cos(\Omega t)\,\hat{i} - \Omega\sin(\Omega t)\,\hat{j} \end{aligned}\right\} \quad (B)$$

but we can use (A) to substitute on the RHS of (B) to get

$$\frac{d}{dt}\hat{i}'(t) = \Omega \hat{j}'(t)$$

$$\frac{d}{dt}\hat{j}'(t) = -\Omega \hat{i}'(t)$$

ie for either unit vector (in fact all basis vectors of x', y', z')

$$\frac{d}{dt}\hat{u}' = \Omega \times \hat{u}' \qquad \left(\vec{\Omega} = \Omega \hat{k} \right)$$

which is what you would actually expect given our first derivation that $\vec{v} = \vec{\omega} \times \vec{r}$.

Now, suppose we write down a general vector in the x, y, z coordinate system, but use the $\hat{i}', \hat{j}', \hat{k}'$ basis

$$\Rightarrow \quad \vec{f} = f_1 \hat{i}' + f_2 \hat{j}' + f_3 \hat{k}'$$

We can now calculate the time derivative of this vector:

$$\left.\frac{d\vec{f}}{dt}\right|_{\text{non-rotating}} = \frac{d}{dt}(f_1 \hat{i}') + \frac{d}{dt}(f_2 \hat{j}') + \frac{d}{dt}(f_3 \hat{k}')$$

$$= \frac{df_1}{dt}\hat{i}' + \frac{df_2}{dt}\hat{j}' + \frac{df_3}{dt}\hat{k}'$$

$$+ f_1 \frac{d\hat{i}'}{dt} + f_2 \frac{d\hat{j}'}{dt} + f_3 \frac{d\hat{k}'}{dt}$$

$$= \left(\frac{d\vec{f}}{dt}\right)_{\text{in rotating frame}} + \vec{\Omega} \times \vec{f}(t)$$

Note Goldstein (Sec 5.5) calls the non-rotating frame the "s" frame for "spatial" and the rotating frame the "b" for "body" frame. You can think of the rotating frame as rotating with the body.

If we let $\vec{f} = \vec{r}$ then

$$\left.\frac{d\vec{r}}{dt}\right|_{space} = \left.\frac{d\vec{r}}{dt}\right|_{body} + \vec{\Omega} \times \vec{r} \qquad \text{———} (c)$$

$$\Rightarrow \quad \vec{v}_s = \vec{v}_{body} + \vec{\Omega} \times \vec{r}$$

If this is not clear, consider a vector $\vec{r}$ that rotates with the rotating axes. Then we must have

$$\left.\frac{d\vec{r}}{dt}\right)_{body} = 0$$

$$\Rightarrow \quad \vec{v}_s = \vec{\Omega} \times \vec{r}$$

ie the velocity is that for circular motion that follows the rotating reference frame.

We can (c) in a more general or operator form

$$\left.\frac{d}{dt}\right|_s = \left.\frac{d}{dt}\right|_b + \vec{\Omega} \times$$

this allows us to look at acceleration

Since $\frac{d^2 \vec{r}}{dt^2} = \vec{a}$

$$\Rightarrow \quad \vec{a}_s = \left[\left(\frac{d}{dt} \right)_b + \vec{\Omega} \times \right] \left[\left(\frac{d\vec{r}}{dt} \right)_b + \vec{\Omega} \times \vec{r} \right]$$

$$\Rightarrow \quad \vec{a}_s = \vec{a}_b + \vec{\Omega} \times \vec{v}_b + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + \frac{d\vec{\Omega}}{dt} \times \vec{r}$$

Many problems will have $\frac{d\Omega}{dt} = 0$ and hence

$$\vec{a}_s = \vec{a}_b + \underbrace{\vec{\Omega} \times \vec{v}_b}_{(1)} + \underbrace{\vec{\Omega} \times (\vec{\Omega} \times \vec{r})}_{(2)}$$

Term (1) is called the Coriolis term & is the origin of the Coriolis force.

Term (2) is called the centrifugal term & is the origin of centrifugal force.

We need one more result to derive Euler's equations. Recall that we separated the forces in a problem according to

$$\dot{\vec{p}}_i = \sum_j \vec{F}_{ij} + \vec{F}_i^{(e)} \qquad \text{———} (\mathrm{I})$$

where $\vec{F}_i^{(e)}$ is an external force, and the $\vec{F}_{ij}$ are internal to the system (ie grav force between particles i & j).

The - ! rate of change of the total momentum of the system is

$$\frac{d}{dt} \sum_i{}' \vec{p}_i = \sum_i{}' F_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}}{}' \vec{F}_{ij} .$$

To relate this to angular momentum, note that

$$\frac{d}{dt}(\vec{L}) = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{v} \times m\vec{v} + \vec{r} \times \frac{d}{dt}(m\vec{v}) = \vec{r} \times \dot{\vec{p}}$$

and take the cross-product of $\vec{r}$ with (I)

$$\Rightarrow \quad \vec{r}_i \times \dot{\vec{p}}_i = \sum_j{}' \vec{r}_i \times \vec{F}_{ij} + \vec{r}_i \times \vec{F}_i^{(e)}$$

If we sum over i, the rate of change of the total angular momentum is then

$$\frac{d}{dt}(\vec{L}) = \sum_i{}' \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = \sum_i{}' \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}}{}' \vec{r}_i \times \vec{F}_{ij}$$

Notice that the last term is actually a sum over pairs:

$$\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}$$

but forces are equal & opposite $\Rightarrow \vec{F}_{ij} = -\vec{F}_{ji}$
thus the sum is really over

$$(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}$$

If the forces between particles lies along their separation $(\vec{r}_i - \vec{r}_j)$ then those cross products vanish.

Hence we find

$$\frac{d}{dt}(\vec{L}) = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} = \vec{N}^{(e)}$$

This is actually a statement of the conservation of angular momentum. $\vec{N}^{(e)}$ represents an external torque on the system, and only if this is non-zero will the <u>total</u> angular momentum of the system change.

Lets now look at the implication of this result in the rotating frame. Since we have shown

$$\left.\frac{d}{dt}\right|_s = \left.\frac{d}{dt}\right|_b + \vec{\Omega} \times$$

we now choose $\vec{\Omega}$ to be $\vec{\omega}$ the rotational velocity of a rotating in body. Then

$$\left.\frac{d\vec{L}}{dt}\right)_s = \left.\frac{d\vec{L}}{dt}\right)_b + \vec{\omega} \times \vec{L} = \vec{N}$$

(dropping e subscript)

Using the alternating symbol* to write down the components, where ϵ_{ijk} is given by

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (3,1,2) \text{ or } (2,3,1) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2) \text{ or } (2,1,3) \\ 0 & \text{otherwise: } i=j \text{ or } j=k \text{ or } k=i \end{cases}$$

[this symbol defines the i^{th} component of a cross-product of two vectors]

* Sometimes called Levi-Civita symbol

Then we can write

$$\left.\frac{dL_i}{dt}\right)_b + \epsilon_{ijk}\,\omega_j L_k = N_i \qquad \left(\begin{array}{c}\text{summation on}\\ j,k \text{ not } i\end{array}\right)$$

If we choose our "body" axes to correspond to the principal axes of the inertia tensor the

$$L_i = I_i \omega_i \qquad (\text{no summation on } i)$$

Since I will be diagonal, hence

$$I_i \frac{d\omega_i}{dt} + \epsilon_{ijk}\,\omega_j I_k \omega_k = N_i$$

writing out explicitly

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1$$

$$I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = N_2$$

$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3$$

Which are known as Euler's Equations of motion for a rigid body. If there is no external torque, then

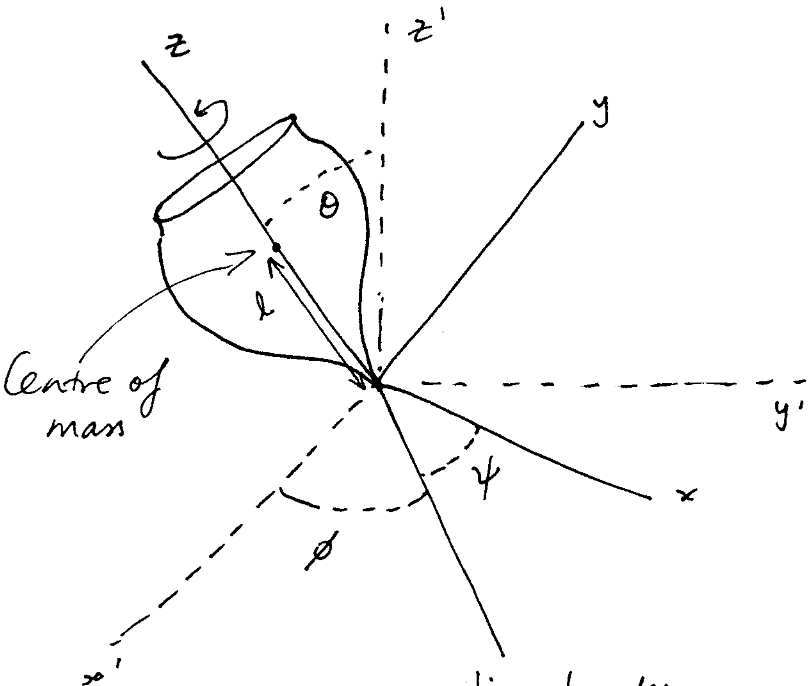
$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3)$$

$$I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1)$$

$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2)$$

The Spinning Top

This is a fascinating & difficult problem but one which we are all familiar with. We begin by establishing the spatial coordinate axes x', y', z' & the body axes x, y, z.



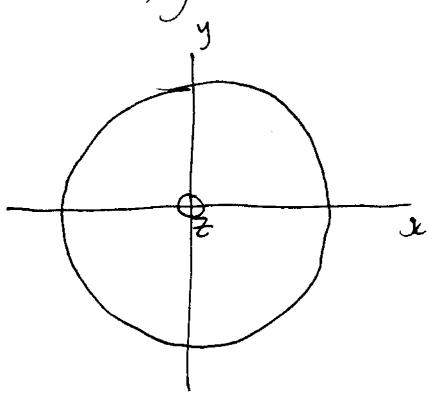
$\dot{\psi}$ = rotation around top's spin axis

$\dot{\phi}$ = precession or rotation of the figure's spin axis around the z' axis

$\dot{\theta}$ = nutation or bobbing up & down of the top's spin axis relative to z'

Usually $\dot{\psi}$ will be the fastest rotation, $\dot{\theta}$ the next fastest, and precession around the vertical, $\dot{\phi}$ the slowest.

The geometry of the top is symmetric about its x, y axes:



The off-diagonal elements of the inertia tensor are

$$I_{jk} = -\int_V \rho \, r_j r_k \, dV$$

where the top has a uniform density so $\rho(\vec{r}) \equiv \rho$.

Because y, x are symmetric about the origin any integrals over them, e.g.

$$I_{xz} = -\int_V \rho\, x z \; dx\, dy\, dz$$

will vanish. Thus $I_{jk} = 0$ & the inertia tensor is diagonal when spinning around the centre of mass.

If we have a diagonal form for I then

$$T = \frac{1}{2}\vec{\omega}\cdot I\vec{\omega} = \frac{1}{2}\left\{ I_{11}\,\omega_1^2 + I_{22}\,\omega_2^2 + I_{33}\,\omega_3^2 \right\}$$

but x & y are symmetric $\Rightarrow I_{11} = I_{22} \overset{def}{=} I_1$

$$\therefore \quad T = \frac{1}{2}\left\{ I_1\omega_1^2 + I_1\omega_2^2 \right\} + \frac{I_3}{2}\omega_3^2 \quad (I_{33} \overset{def}{=} I_3)$$

$$= \frac{I_1}{2}\left\{ \omega_1^2 + \omega_2^2 \right\} + \frac{I_3}{2}\omega_3^2$$

We next want to write down the components of $\vec{\omega}$ in terms of the Euler angles we can measure in the spatial frame.
$\vec{\omega}$ can be written as a sum of components in the $\vec{\omega}_\phi$, $\vec{\omega}_\theta$, $\vec{\omega}_\psi$ directions & without proof we state (see Goldstein 4.9)

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

Note that
$\vec{\omega}_\psi$ (with component $\dot\psi$) is along the z-axis of the body
$\vec{\omega}_\phi$ (" " $\dot\phi$) " " " z'-axis of the spatial frame
$\vec{\omega}_\theta$ (" " $\dot\theta$) is along the line of nodes.

Calculation of ω_1^2, ω_2^2 & ω_3^2 then gives

$$T = \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot\psi + \dot\phi\cos\theta\right)^2$$

The potential energy of the system will be given by the change in position of the centre of mass

$$\therefore \; V = mgl\cos\theta \qquad \left(\text{taking zero point at } z'=0\right)$$

Hence the Lagrangian is

$$L = \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + \frac{I_3}{2}\left(\dot\psi + \dot\phi\cos\theta\right)^2 - Mgl\cos\theta$$

Rather than going straight to the E-L equations we first look for cyclic coordinates & conserved quantities.

Since ψ & ϕ do not appear explicitly, ie $\frac{\partial L}{\partial \psi} = \frac{\partial L}{\partial \phi} = 0$ then we have

$$\frac{\partial L}{\partial \dot\psi} \stackrel{\text{def}}{=} p_\psi = I_3\left(\dot\psi + \dot\phi\cos\theta\right) \quad \& \quad \frac{d}{dt}(p_\psi) = 0.$$

$$= I_3\,\omega_3$$

ie the z-component of the angular momentum in the body is constant also

$$\frac{\partial L}{\partial \dot\phi} \stackrel{\text{def}}{=} p_\phi = I_1 \dot\phi \sin^2\theta + I_3 \cos\theta(\dot\psi + \dot\phi\cos\theta)$$

$$= (I_1 \sin^2\theta + I_3 \cos^2\theta)\dot\phi + I_3 \cos\theta\, \dot\psi$$

and $\frac{d}{dt}(p_\phi) = 0$

ie the component of the angular momentum around the spatial z'-axis is constant.

Since p_ψ, p_ϕ are constants we can write

$$p_\psi = I_1 a$$
$$p_\phi = I_1 b$$

also note that $\omega_3 = \frac{p_\psi}{I_3} = \frac{I_1 a}{I_3}$ is a constant.

We can also apply conservation of energy here (there are no external torques/forces). Thus

$$E = T + V = \frac{I_1}{2}(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{I_3}{2}\omega_3^2 + Mgl\cos\theta$$

is also a constant.

We can now apply those results. Firstly since

$$p_\psi = I_3(\dot\psi + \dot\phi\cos\theta) = I_1 a$$

$$\Rightarrow \quad I_3 \dot\psi = I_1 a - I_3 \dot\phi \cos\theta \quad \text{—— (A)}$$

then by substituting this result in

$$p_\phi = (I_1 \sin^2\theta + I_3 \cos^2\theta)\dot{\phi} + I_3 \dot{\psi} \cos\theta = I_1 b$$

we can eliminate $\dot{\psi}$ & arrive at

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \qquad \text{———} \quad (B)$$

Hence if we knew $\theta(t)$ we could find $\phi(t)$ by integrating w.r.t. t. We can also back substitute (B) into (A) to give $\dot{\psi}$:

$$\dot{\psi} = \frac{I_1 a}{I_3} - \cos\theta\left(\frac{b - a\cos\theta}{\sin^2\theta}\right)$$

& again if we knew $\theta(t)$ we could then find $\psi(t)$.

To progress toward finding $\theta(t)$, let us now look at the energy equation. Since we showed ω_3 is a constant then we can write

$$E - \frac{I_3 \omega_3^2}{2} \stackrel{\text{def}}{=} E' = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + Mgl\cos\theta$$

where E' must also be constant since both E & I_3, ω_3 are constants.

Substituting for $\dot{\phi}^2$ using (B) gives

$$E' = \frac{I_1}{2}\dot{\theta}^2 + \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

which is an equation in $\underline{\theta \text{ only}}$. While it looks complex, we can write it in a form

$$E' = \frac{I_1}{2}\dot{\theta}^2 + V'(\theta) \quad \text{———} \quad (C)$$

where $V'(\theta) = Mgl\cos\theta + \frac{I_1}{2}\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2$

Then this looks like a 1-dimensional problem with a rather complex potential term.

If we define $\alpha = \frac{2E - I_3\omega_3^2}{I_1}$ & $\beta = \frac{2Mgl}{I_1}$

Then (C) can be written as

$$\alpha = \dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{\sin^2\theta} + \beta\cos\theta \quad \text{———} \quad (D)$$

and we have 4 constants associated with the motion, a, b, α, β.

(D) can be put into a slightly simpler form by writing $u = \cos\theta$. Then $\dot{u} = -\dot{\theta}\sin\theta \Rightarrow \dot{\theta} = \frac{-\dot{u}}{\sqrt{1-u^2}}$ and

$$\alpha = \frac{\dot{u}^2}{(1-u^2)} + \frac{(b-au)^2}{1-u^2} + \beta u$$

$$\Rightarrow \quad \dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b-au)^2$$

$$\Rightarrow \quad \frac{du}{dt} = \sqrt{(1-u^2)(\alpha-\beta u) - (b-au)^2} \quad \text{———} \quad (E)$$

We would then integrate to get

$$t = \int_0^t dt = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha-\beta u)-(b-au)^2}}$$

Similarly substituting for $\cos\theta = u$ in $\dot\phi = \dfrac{b-a\cos\theta}{\sin^2\theta}$

$$\Rightarrow \quad d\phi = \frac{b-au}{1-u^2}\,\frac{du}{\sqrt{(1-u^2)(\alpha-\beta u)-(b-au)^2}}$$

and for $\dot\psi$

$$d\psi = \left(\frac{I_1 a}{I_3} - \frac{u(b-au)}{1-u^2}\right)\frac{du}{\sqrt{(1-u^2)(\alpha-\beta u)-(b-au)^2}}$$

While such integrals can sometimes be done using elliptic functions the results are very messy & uninformative. We can, however, tell a great deal about the motion without doing them.

The key part of discussing the behavior of the system is to go back to (E), which is a cubic equation in $u = \cos\theta$.

Let us write

$$\dot u^2 = \beta u^3 - (\alpha + a^2)u^2 + (2ab-\beta)u + (\alpha - b^2) \stackrel{\text{def}}{=} f(u)$$

Clearly the value of $f(u)$ will determine the evolution of $\dot u$ and thus $u(t)$.

Let us look first at the possible values for α & β. Since $\beta = \frac{2Mgl}{I_1}$ this must be greater than zero.

For $\alpha = \frac{2E - I_3 \omega_3^2}{I_1}$ this clearly depends upon $2E - I_3 \omega_3^2$. Since subtracting $I_3 \omega_3^2$ from $2E$ leaves

$$I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + 2Mgl\cos\theta$$

Then provided $\theta < \frac{\pi}{2}$ i.e. $\cos\theta > 0$ then this must produce $\alpha > 0$. Note that if the top spun on a stand that allowed $\cos\theta < 0$ then α would become negative.

The roots of $f(u)$ also delineate different areas of behaviour since they determine where $f(u)$ changes from +ve to -ve. The values of θ for which these changes occur are called "turning angles."

Classification of Top behavior

We have seen that the behavior of θ is determined by a cubic equation in $\cos\theta$.
Cubic equations have the following possibilities for roots

(a) one real, two complex roots
(b) three real roots, two equal
(c) three real & unequal roots

For the $f(u)$ cubic, the exact case we have for the top will depend on the constants a, b, α, β.

It is also important to note that since $u = \cos\theta$ $-1 < u \leq 1$. Further, since we have assumed the top is spinning above a flat plane $\cos\theta > 0$ and we consider solutions for $u > 0$.
Since

$$f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2$$
$$= \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2)$$

at large u we have

$$f(u) \simeq \beta u^3 \qquad \text{since } \beta > 0, \ f(u) \text{ large \& +ve}$$

Similarly for large $-$ve u we have

$$f(u) \simeq \beta u^3 \qquad f(u) \text{ large \& -ve}$$

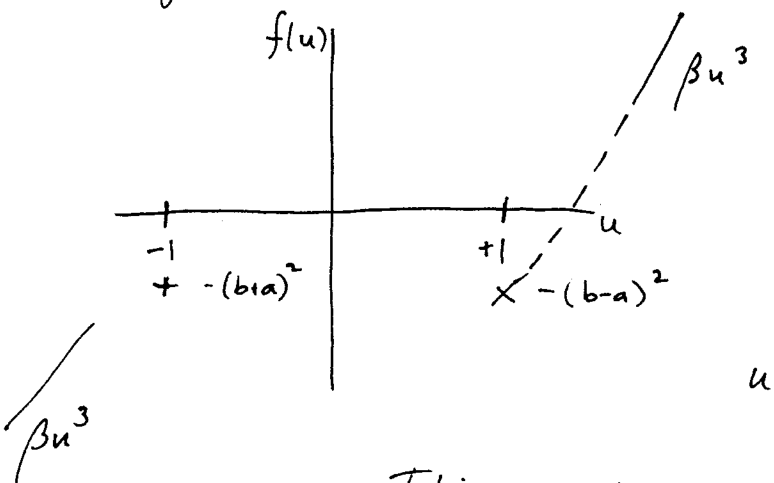
Using the factorized form of $f(u)$ we see that at $u^2 = 1$, ie $u = \pm 1$ then

$$f(u) = -(b \mp a)^2 \quad \text{for } \pm 1 = u$$

this must always be negative or zero.

If $b = a$ then $p_\psi = \frac{a}{I_1} = p_\phi$ and so the top must be standing vertically.

Lets draw the $f(u)$ plane & add the information we have gathered so far:

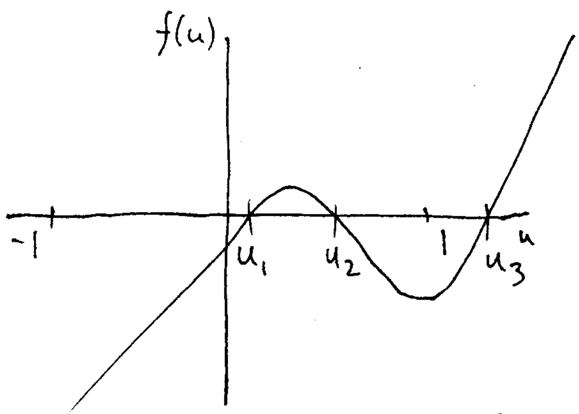


Since we know that $-(b-a)^2$ must be $-$ve & βu^3 is $+$ve there must be a root between $u = 1$ & ∞.

This root cannot correspond to a real angle since $u > 1$.

Notice physical motion can only occur when $\dot{u}^2 = f(u) > 0$ (otherwise $\dot{u}$ would be imaginary).

Since there must be some $+$ve value to $f(u)$ in between -1 & 1 then we must have two roots between -1 & 1. Also since $u > 0$ for the top on the plane we expect two solutions with $u > 0$ (but this need not be the case if the top was supported on a tall pivot)



The precise values of u_1, u_2 & the value of θ will then determine $\dot{\phi}, \dot{\psi}$.

It is normal to trace the motion of the axis of the top on a sphere using θ, ϕ coordinates.

Since physical solutions require $f(u) > 0$ then we have limiting values for u of u_1 & u_2. This means that θ is bounded to lie between

$$\arccos u_2 < \theta < \arccos u_1$$

What happens to $\dot{\theta}$ at u_1 & u_2? $\dot{\theta} = 0$

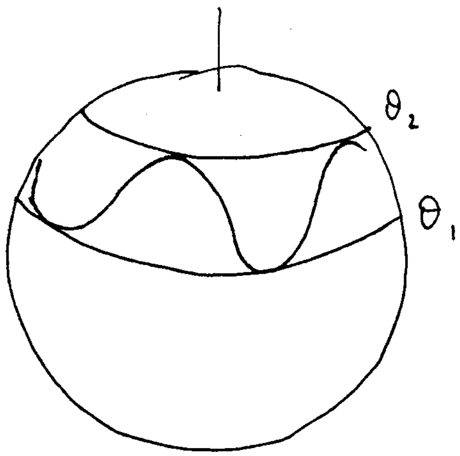
Next, since $\dot{\phi} = \frac{b - au}{1-u^2}$ & $\dot{\psi} = \frac{I_1 a}{I_3} - \frac{u(b-au)}{1-u^2}$

much of the resulting behavior is determined by $b-au$ and specifically the value of the root $b - au = 0$. If we denote the root u', then

$$u' = \frac{b}{a}$$

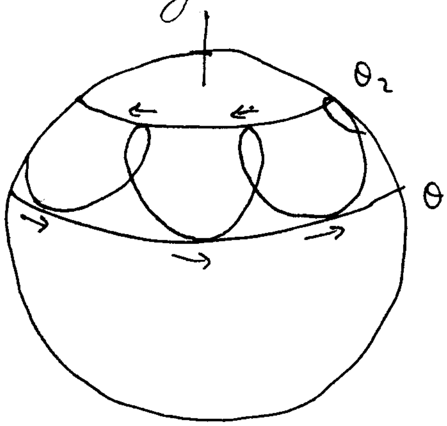
will determine much of the overall shape of the locus.

Suppose $u' > u_2$, then $\forall\, u < u_2$ $b-au$ must have the same sign. Thus $\dot{\phi}$ must always have the same sign. Therefore the motion must look as follows:



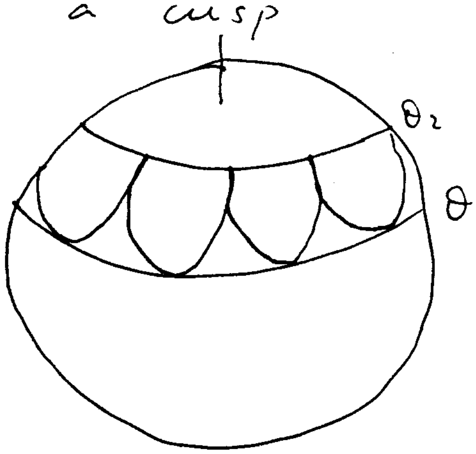
ϕ always increases in one direction or another, so the top precesses, while it nutates between θ_2 & θ_1.

If u' is such that it lies between u_1 & u_2 then the behavior of $\dot{\phi}$ can include both forward & backward movement. However, the average $\dot{\phi}$ will not vanish so there is still a net precession. Motion will generally look as follows:



The top axis will momentarily loop backwards before beginning another forward movement.

Lastly, if u' equals u_1 or u_2 then both $\dot{\phi}$ & $\dot{\theta}$ vanish at θ_1 or θ_2. This produces a cusp



This is actually a fairly common motion. If we set up a top initially with

$$\theta = \theta_0$$
$$\dot{\theta} = 0$$
$$\dot{\phi} = 0$$

and then release it, we are effectively starting from the cusp of one of these paths. In this case the starting angle θ_0 must describe one of the roots, specifically the upper angle. Note that the top has to fall from this initial configuration since by conservation of energy as $\dot{\theta}$ & $\dot{\phi}$ rise the potential energy must go down & hence $\cos\theta$ becomes smaller.

We can make more quantitative predictions if we consider a "fast top" where

$$\frac{1}{2} I_3 \omega_3^2 \gg 2Mgl$$

in this case both nutation & precession will be small perturbations compared to the rotation of the top's axis. The factor of 2 on the Mgl comes from the maximum possible change in the P.E.

Suppose initially the top is at an angle θ_0 & $\cos\theta_0 = u_0$. The nutation can evolve to a second point u_1 corresponding to the other physical root of $f(u)$.

Since we start initially from rest, ie $\dot{\theta} = 0$ this means $b - au_0 = 0$ initially. Thus the energy equation reduces to

$$E' = Mgl\cos\theta_0 \quad \Rightarrow \quad \alpha = \beta u_0$$

after substituting for E' & Mgl with α & β

Given $b - au_0 = 0$ & $\alpha = \beta u_0$ $f(u)$ can be rewritten

$$f(u) = (u_0 - u)\left[\beta(1-u^2) - a^2(u_0 - u)\right]$$

hence the roots of $f(u)$ (other than u_0) are given by the quadratic in the [ ]. Therefore the other desired root u_1 is given by

$$(1-u_1^2) - \frac{a^2}{\beta}(u_0 - u_1) = 0 \qquad \text{—— (A)}$$

if we let $x_1 = u_0 - u_1 \Rightarrow u_1^2 = u_0^2 - 2x_1u_0 + x_1^2$ then (A) can be written as

$$x_1^2 + \left\{\frac{a^2}{\beta} - 2u_0\right\}x_1 - \left\{1 - u_0^2\right\} = 0$$

$$\text{or} \quad x_1^2 + px_1 - q = 0 \qquad \text{(B)}$$

where

$$p = \frac{a^2}{\beta} - 2\cos\theta_0 \qquad q = \sin^2\theta_0$$

What can we say about the values of p & q? Clearly $0 \leq q \leq 1$, and for p expand $\frac{a^2}{\beta}$:

$$a = \frac{p_\psi}{I_1} = \frac{I_3\omega_3}{I_1}$$

$$\Rightarrow \frac{a^2}{\beta} = \frac{I_3^2\omega_3^2}{I_1^2}\frac{I_1}{2Mgl} = \left(\frac{I_3}{I_1}\right)\frac{I_3\omega_3^2}{2Mgl}$$

Unless $I_3 \ll I_1$ (ie the top is a very long & thin cigar!) then the fast top condition

$$\tfrac{1}{2}I_3\omega_3^2 \gg 2Mgl$$

ensures $\frac{a^2}{\beta}$ is large & thus $p \gg q$.

For $p \gg q$ we can write the general solution to (8) as follows:

$$x = -\frac{p}{2} \pm \frac{p}{2}\left(1 + \frac{4q}{p^2}\right)^{1/2}$$

expand square root in powers of $\frac{q}{p^2}$ since this is small

$$\Rightarrow \quad x_1 = -\frac{p}{2} \pm \frac{p}{2}\left\{1 + \frac{1}{2}\frac{4q}{p^2} + \dots\right\}$$

$$\simeq -\frac{p}{2} \pm \frac{p}{2}\left(1 + \frac{2q}{p^2}\right)$$

Since x_1 must be small & p is large,

$$\Rightarrow \quad x_1 \simeq \frac{p}{2} \times \frac{2q}{p^2} = \frac{q}{p}$$

is the only physically realizable solution.

If we neglect the factor of $2\cos\theta_0$ in p, then this can be written

$$x_1 = \frac{\beta \sin^2\theta_0}{a^2} = \frac{I_1}{I_3}\frac{2Mgl}{I_3\omega_3^2}\sin^2\theta_0$$

This demonstrates a key result – the faster the top is spun, so that ω_3^2 increases, the smaller $x_1 = u_0 - u_1$ & thus the smaller the nutation.

A similar analysis shows that the angular frequency of nutation between θ_0 & θ_1 is

$$a = \frac{I_3}{I_1}\omega_3$$

which <u>increases</u> the faster the top is spun.