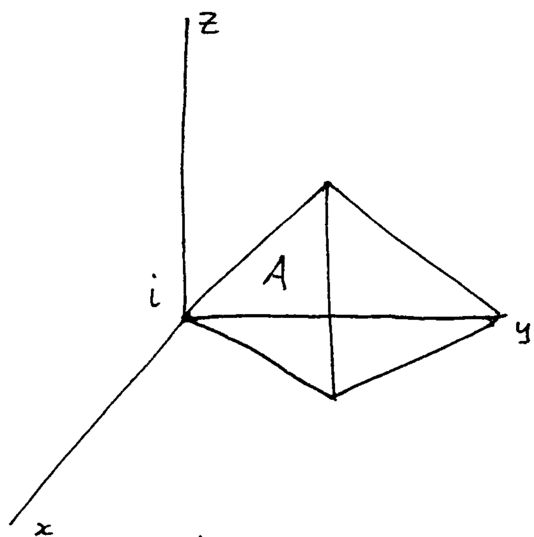
Euler Angles

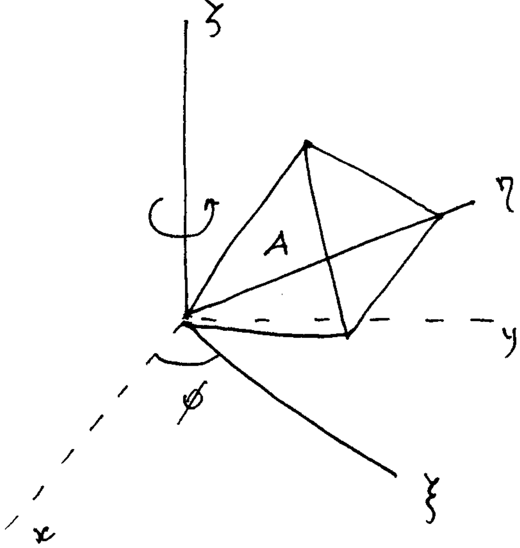
We've seen that the 3×3 matrices representing rotations have their 9 degrees of freedom reduced to 3 by the six equations in the orthogonality condition. What are those degrees of freedom?

Recall from our initial discussion of rigid body kinematics that we can think of rotations as being equivalent to establishing a point on a sphere followed by a rotation. Let's express this using matrices.



Have a body with point i at the origin.

Step 1: rotate counterclockwise around the z-axis, specify this with an angle $\phi \in [0, 2\pi]$



We now have new set of axes

ζ = "new" z

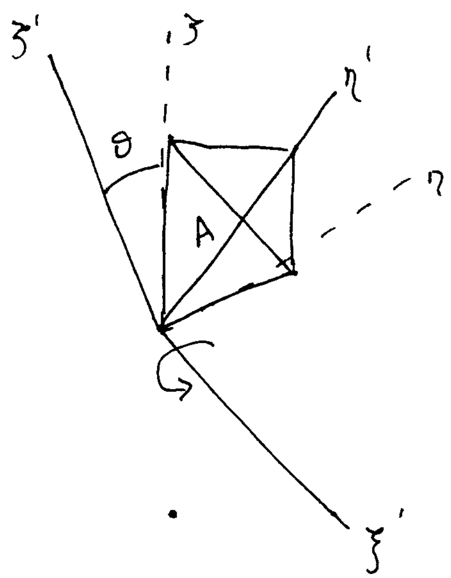
η = new y

ξ = new x

The transformation can be represented by the matrix

$$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

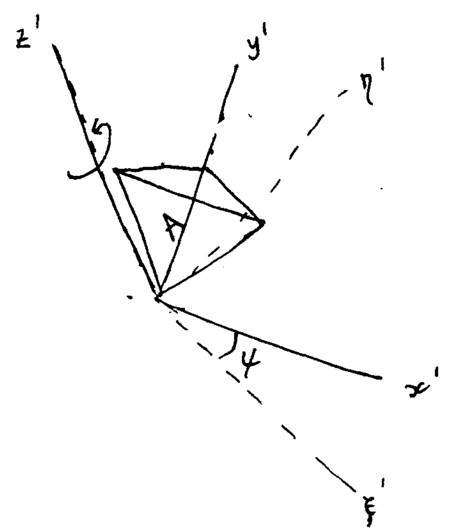
The next step is to apply a rotation about the ξ axis, counterclockwise & specified by angle $\theta \in [0, \pi]$. The axes map to ξ', η', ζ'



Since ξ is the effective x-axis of ξ, η, ζ we describe this rotation by a matrix C where

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

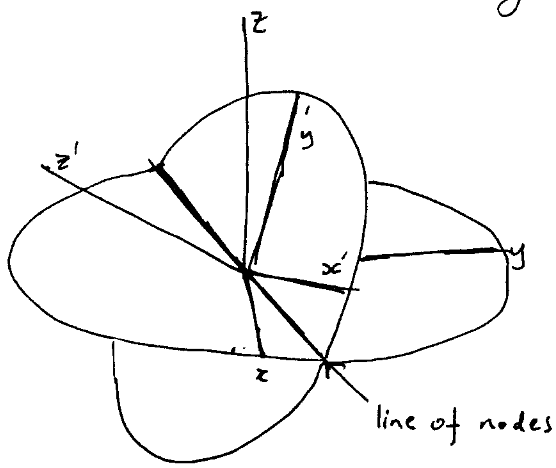
The last rotation is one about the new ζ' axis. This will be through an angle $\psi \in [0, 2\pi]$



This describes a final set of axes, x', y', z' & the rotation matrix is

$$B = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The intersection of the x', y' planes & x, y planes is called the 'line of nodes'.



Given the matrices BCD we can multiply them out to get the matrix for the transformation, ie if $A = BCD$ then

$$A = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

Further, since A is also an orthogonal transformation we know that

$$A^{-1} = A^{T}$$

It is worth noting that the sequence of rotations is quite arbitrary. For example we could have chosen any of the x, y, z axes for the first rotation. The definition we have discussed is called the x-convention but a different one is used in quantum mechanics & yet another one in aerodynamics.

Euler angles also have a problem when x', y' & x, y are the same (z & z' are either aligned or anti-aligned). Then the rotation is only one by $\phi + \psi$, (or $\phi - \psi$) there are then infinitely many transformations that map to the same point. This has real implications for assemblies that pivot in 3D "gimbal lock" (google gimbal lock apollo!)

Inertia Tensor

To date we have not considered rotational kinetic energy in any detail. We now address this issue.

Recall that rotational motion with angular speed ω, about an axis given by $\vec{\omega}$, the velocity vector will be

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Since angular momentum is defined by

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$$

then for rotational motion

$$\vec{L} = m\vec{r} \times (\vec{\omega} \times \vec{r})$$

Further, if we consider this for a rigid body with components labelled i, then we have

$$\vec{L} = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

Using the vector identity

$$\vec{c} \times (\vec{a} \times \vec{b}) = (\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b} \quad \text{we get}$$

$$\vec{L} = \sum_i (m_i r_i^2 \vec{\omega} - m_i (\vec{\omega} \cdot \vec{r}_i)\vec{r}_i)$$

writing out the L_x component:

$$L_x = \sum_i (m_i r_i^2 \omega_x - m_i x_i (\omega_x x_i + \omega_y y_i + \omega_z z_i))$$

$$= \omega_x \sum_i m_i (r_i^2 - x_i^2) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i$$

We also have similar expressions for L_y & L_z. Notice that L_x is a linear function of all the ω components ($\omega_x, \omega_y, \omega_z$). The same is true for L_y, L_z.

Hence let us define components of a matrix I by

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

The on-diagonal elements are defined by

$$I_{jj} = \sum_i m_i (r_i^2 - (x_i)_j^2)$$

no summation on j
$j = 1, 2, 3$ for the x, y, z components

These elements are called the <u>moment of inertia coefficients.</u> While the off-diagonal elements are defined by

$$I_{jk} = -\sum_i m_i (x_i)_j (x_i)_k \qquad \begin{matrix} j,k = 1,2,3 \\ \text{but } j \neq k \end{matrix}$$

these off-diagonal elements are known as the <u>products of inertia.</u>

If the rigid body is described by a volume then the sum over the components can be replaced by an integral over the volume ~~of~~ & density of the object:

$$\sum_i m_i \longrightarrow \int_V \rho(\vec{r})\, dV$$

hence, for example,

$$I_{xx} = \int_V \rho(\vec{r})\,(r^2 - x^2)\; dV$$

and for all elements of I_{jk}

$$I_{jk} = \int_V \rho(\vec{r})\,(r^2 \delta_{jk} - r_j\, r_k)\, dV \quad k,j = 1,2,3$$

In compact notation we can write

$$\vec{L} = I\,\vec{\omega}$$

where I is a matrix, and is called the <u>moment of inertia tensor.</u>

Why tensor? (Goldstei 5.2)

Tensors are actually quite a straightforward idea (with a fancy name!)
Conside a 3d vector $\vec{r} = (x, y, z) \equiv x_i$
The vector has three components, which are labelled by one index i.
- a vector is essentially a tensor of rank 1 (it has one index)

Conside a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$

This has nine components labelled by two indices i & j.
- a tensor of rank 2 can be represented as a matrix

A tensor of rank 3 would be written c_{ijk}
It would have $3 \times 3 \times 3 = 27$ components

Thus a tensor of rank n is written $c_{i_1 \ldots i_n}$ and would have 3^n components.

The key issue about tensors is that they transform linearly in their components under coordinate changes. Consider the following:

Suppose a_{ij} defines a coordinate change
Then a vector transforms as

$$x_i' = a_{ij} x_j \qquad \text{(summation implied)}$$

a tensor of rank 2 will transform as

$$T'_{ij} = a_{ik}\, a_{jl}\, T_{kl}$$

a rank 3 tensor would transform as

$$R'_{ijk} = a_{i\cancel{k}n}\, a_{jl}\, a_{km}\, R_{\cancel{k}n\,lm}$$

and so on. Tensors are really defined in terms of their transformation properties under coordinate changes (in our current discussion orthogonal coordinate changes). While a matrix can be transformed under many different kinds of transformations.

General Form of I

Since we have shown the diagonal elements are

$$I_{jj} = \sum_i m_i \left(\vec{r}_i^{\,2} - (x_i)_j^2\right) \qquad j = 1,2,3$$

and

$$I_{jk} = -\sum_i m_i\, (x_i)_j\, (x_i)_k \qquad j = 1,2,3,\; k = 1,2,3 \text{ but } j \neq k$$

we can combine the two results into

$$I_{jk} = \sum_i m_i \left(\delta_{jk}\, \vec{r}_i^{\,2} - (x_i)_j (x_i)_k\right) \quad \forall\; j,k = 1,2,3$$

which is equivalent to the integral version

$$I_{jk} = \int_V \rho(\vec{r})\left(\vec{r}^{\,2}\delta_{jk} - r_j\, r_k\right) dV$$ given earlier.

Remember $\vec{r}_i^{\,2} \equiv (x_i)_j (x_i)_j$.

More on the Inertia Tensor

Thus far our study of the kinetic energy of rotation has been restricted to a hoop, where

$$T_{Rotation} = \frac{1}{2} m v_{rot}^2$$

where v_{rot} is the rotational velocity of the mass on the edge of the hoop. In general for rotational motion about a point

$$T_{Rot} = \sum_i \frac{1}{2} m_i v_i^2$$

where the sum is over components or parts of a system.

Since for rotational motion $\vec{v}_i = \vec{\omega} \times \vec{r}$

$$T = \sum_i \frac{1}{2} m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i)$$

Using the vector identity $\vec{a} \cdot (\vec{L} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

$$\Rightarrow \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i) = \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i)$$

thus

$$T = \sum_i \frac{\vec{\omega}}{2} \cdot (\vec{r}_i \times m_i \vec{v}_i) = \frac{\vec{\omega}}{2} \cdot \sum_i \vec{r}_i \times m \vec{v}_i$$

$$= \frac{1}{2} \vec{\omega} \cdot \vec{L}$$

$$= \frac{1}{2} \vec{\omega} \cdot I \vec{\omega} \qquad \text{since } \vec{L} = I \vec{\omega}$$

$$= \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta \qquad \text{using summation convention.}$$

If we write $\vec{\omega} = \omega \hat{n}$, where $\hat{n}$ is the unit vector along the axis of rotation

$$\Rightarrow \quad T = \frac{1}{2}\omega\hat{n}.I\omega\hat{n} = \frac{\omega^2}{2}\hat{n}.I\hat{n}$$

Writing out using the summation convention on α, β

$$T = \frac{\omega^2}{2} n_\alpha I_{\alpha\beta} n_\beta \qquad \text{but } I_{\alpha\beta} = \sum_i m_i \left(\delta_{\alpha\beta}\vec{r}_i^{\,2} - (r_i)_\alpha (r_i)_\beta\right)$$

thus

$$T = \frac{\omega^2}{2} \cdot \sum_i m_i \left(\delta_{\alpha\beta} n_\alpha n_\beta \vec{r}_i^{\,2} - (r_i)_\alpha (r_i)_\beta n_\alpha n_\beta\right)$$

but $\delta_{\alpha\beta} n_\alpha n_\beta = n_\alpha n_\alpha = \hat{n}^2 = 1$

and $(r_i)_\alpha (r_i)_\beta n_\alpha n_\beta = (\vec{r}_i.\hat{n})(\vec{r}_i.\hat{n}) = (\vec{r}_i.\hat{n})^2$

Therefore

$$T = \frac{\omega^2}{2} \cdot \sum_i m_i \left(\vec{r}_i^{\,2} - (\vec{r}_i.\hat{n})^2\right)$$

$$\equiv \frac{I\omega^2}{2}$$

where by definition $I = \sum_i m_i \left(\vec{r}_i^{\,2} - (\vec{r}_i.\hat{n})^2\right)$ which is a scalar value known as the <u>moment of inertia</u> about the axis of rotation. The integral form is

$$I = \int_V \rho(\vec{r})\left(\vec{r}^{\,2} - (\vec{r}.\hat{n})^2\right) dV$$

we can calculate this for different volumes e.g. cube, cylinder etc using volume integral approach (see homework 4).

Clearly the inertia tensor is symmetric: $I_{\alpha\beta} = I_{\beta\alpha}$ which means that it has only six independent components (three on the diagonal & 3 off-diagonal). This is still a somewhat awkward (at least computationally expensive) matrix. It would be helpful if we could write I in a diagonal form:

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

then we would find

$$\begin{aligned} T &= \frac{1}{2} \vec{\omega} \cdot I \vec{\omega} \\ &= \frac{1}{2} \vec{\omega} \cdot (I_1 \omega_1, \; I_2 \omega_2, \; I_3 \omega_3) \\ &= \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \end{aligned}$$

Notice that the components of I depend on the orientation & position of the body relative to chosen axes. Changing the coordinate system will change I. Hence diagonalizing I is really about finding a coordinate transformation that brings I into a diagonal form.

How does I transform under coordinate changes?

For a transformation A, we have $\vec{\omega}' = A\vec{\omega}$ and for angular momentum $\vec{L}' = A\vec{L}$

Using $\vec{L} = I\vec{\omega}$ we get

$$\vec{L}' = A\vec{L} = (I\vec{\omega})' = A(I\vec{\omega}) \overset{\text{def}}{\equiv} I'\vec{\omega}'$$

but since $\vec{\omega}$ transforms as $\vec{\omega}' = L\vec{\omega}$

$$\Rightarrow \quad \vec{L}' = A(I\vec{\omega}) = AI(A^{-1}A)\vec{\omega}$$
$$= (AIA^{-1})A\vec{\omega}$$
$$= I'\vec{\omega}'$$
$$\therefore \quad I' = AIA^{-1}$$

Hence we have shown under a coordinate change
$I' = AIA^{-1}$,
for rotations A is an orthogonal matrix.

To find A we need to use matrix algebra. Recall that for a general matrix M, the eigenvalues are defined by the equation

$$M\vec{v} = \lambda\vec{v}$$ where $\vec{v}$ is an eigenvector & λ is an eigenvalue

$$\Rightarrow \quad (M - \mathbb{1}\lambda)\vec{v} = \vec{0}$$

Assuming the matrix M is non-singular (has an inverse) then we can find the λ by solving

$$\det(M - \mathbb{1}\lambda) = 0$$

Once the eigenvalues have been found we then solve for the eigenvectors by Gaussian elimination on

$$(M - \lambda\mathbb{1})\vec{v} = \vec{0}$$

As an example, consider a 2d matrix $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

This has eigenvalues $\lambda_1 = 1$; $\lambda_2 = 3$. The eigenvectors are found by solving

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

&

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Hence if we now take $M = I$, then we need to solve

$$\det \begin{pmatrix} I_{xx} - \lambda & I_{xy} & I_{zx} \\ I_{xy} & I_{yy} - \lambda & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} - \lambda \end{pmatrix} = 0$$

this gives the eigenvalues, Then we solve

$$\begin{pmatrix} I_{xx} - \lambda & I_{xy} & I_{zx} \\ I_{xy} & I_{yy} - \lambda & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$$

for the eigenvectors. Once we have the eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ we can diagonalize I as follows:

Construct a matrix P where

$$P = \left[\begin{pmatrix} \\ \vec{v}_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ \vec{v}_2 \\ \\ \end{pmatrix} \begin{pmatrix} \\ \vec{v}_3 \\ \\ \end{pmatrix} \right]$$

$$\Rightarrow IP = \left[\begin{pmatrix} \\ I\vec{v}_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ I\vec{v}_2 \\ \\ \end{pmatrix} \begin{pmatrix} \\ I\vec{v}_3 \\ \\ \end{pmatrix} \right] = \left[\begin{pmatrix} \\ \lambda_1 \vec{v}_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ \lambda_2 \vec{v}_2 \\ \\ \end{pmatrix} \begin{pmatrix} \\ \lambda_3 \vec{v}_3 \\ \\ \end{pmatrix} \right]$$

and hence

$$IP = \left[\begin{pmatrix} \\ \vec{v}_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ \vec{v}_2 \\ \\ \end{pmatrix} \begin{pmatrix} \\ \vec{v}_3 \\ \\ \end{pmatrix} \right] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$= P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Multiply by P^{-1}, then

$$P^{-1} I P = P^{-1} P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \stackrel{\text{def}}{\equiv} I_{\text{diagonal}} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Thus the matrix of eigenvectors defines a matrix P that can be used to transform I into a diagonal form. In this diagonal form the I_i (eigenvalues of I) are called the components of the principle moment of inertia tensor.

P will correspond to a rotation through the Euler angles. It will map x, y, z to x', y', z' and the direction of x', y', z' are called the principle axes (and correspond to the eigenvectors of I.)

I again emphasize that the components of I are completely dependent upon the coordinate system used!