PHYS3300 - Assignment 5

Due in class Mon, Dec 4th, 2023 (Time allowed=two weeks)

Show all working to receive full credit, especially when performing integrals via substitution. If it is acceptable to quote a standard result then it will be mentioned in the question.

(1) (a) **(4 marks)** Using the components of ω_1, ω_2, ω_3, as given on the bottom of page 90 in the notes, show that the kinetic energy of the top is

$$T = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2.$$

(b) **(6 marks)** On pages 101-103 of the notes we showed how the extent of the nutation of the top (as measured by x_1) goes down with increasing spin frequency of the top. We can also calculate the frequency of the nutation. Show the $f(u)$ polynomial,

$$f(u) = (u_0 - u)\left[\beta(1-u^2) - a^2(u_0 - u)\right]$$

becomes

$$f(u) = a^2 x(x_1 - x) = \dot{x}^2$$

where $x = u_0 - u$, $x_1 = u_0 - u_1$. Using a substitution $y = x - x_1/2$ derive an equation in y describing the nutation around the midpoint $x_1/2$. If $x = 0$ at $t = 0$ give the solution for x and what is the frequency of the nutation?

(c) **(6 marks)** Suppose that the nutation angle is fixed, such motion is called regular precession, so that $\theta_1 = \theta_2 = \theta_0$. In this case the $f(u)$ polynomial has two roots at zero, implying that

$$\frac{df}{du} = 0,$$

and $u = u_0$ always. Show that these conditions lead to

$$Mgl = \dot{\phi}(I_3\dot{\psi} - (I_1 - I_3)\dot{\phi}\cos\theta_0),$$

note definitions for a and β can be found in the notes.

(d) **(4 marks)** Show that for this equation to describe a real valued $\dot{\phi}$ we must have that

$$I_3^2\omega_3^2 > 4MglI_1\cos\theta_0.$$

(Note that for $\theta_0 > \pi/2$ then any ω_3 can lead to uniform precession.)

(2) A bead, of mass m, is allowed to roll inside a fixed sphere of radius R.

(a) **(4 marks)** Find the equations of motion using the Lagrangian formalism.

(b) **(6 marks)** Find the equations of motion using the Hamiltonian formalism.

(3) Suppose a bead, mass m, moves on a wire, without friction. The wire is bent into the shape of a helix with cylindrical polar coordinates given by (r, ϕ, z). By virtue of being a helix, the z-coordinate is given by $z = c\phi$, and the radial coordinate $r = R$, where both c and R are constants. In this question assume that the vertical axis corresponds to the z-axis.

(a) **(3 marks)** What is the generalized coordinate in this case? Give both the K.E. and P.E. of the bead as it moves along the wire under gravity.

(b) **(3 marks)** Find the conjugate momentum associated with the generalized coordinate and hence evaluate the Hamiltonian of the system.

(c) **(2 marks)** Write down Hamilton's equations for this system. Now find the equation for $\ddot{\phi}$ in terms of

g, c, R, and hence for $\ddot{z}$. Give your answer for $\ddot{z}$ in terms of the angle α where $\tan\alpha = c/R$. This relationship means that as the bead moves through an angle $\phi = 2\pi$ then it also rises or falls a distance $2\pi c$.
(d) **(2 marks)** What happens when R=0? Discuss this result in terms of what you know about a bead falling along a vertical wire.

(4) A particle of mass m is placed on a frictionless plane with an incline angle α. The angle changes over time at a constant angular rate given by ω and $\alpha = 0$ at $t = 0$. As the angle becomes larger and larger so the acceleration of the particle will increase.
(a) **(4 marks)** Determine the equation of motion using the Lagrangian approach.
(b) **(4 marks)** Determine the Hamiltonian and derive Hamilton's equations of motion.
(c) **(2 marks)** Is the Hamiltonian equal to the total energy? Is the total energy conserved? Explain.