PHYS3300 - Assignment 4

Due Wed Nov 22, 2023 (Time allowed=two weeks + fall break)

Show all working to receive full credit, especially when performing integrals via substitution. If it is acceptable to quote a standard result then it will be mentioned in the question.

(1) Consider a body that is described by both an inertial (spatial) coordinate frame, and a body frame that rotates with angular velocity $\boldsymbol{\omega}$.
(a)**(3 marks)** Show that when described in terms of the body frame velocity $\mathbf{v}_b$ the kinetic energy is given by

$$T = \frac{m}{2}\left[{\mathbf{v}_b}^2 + 2\mathbf{v}_b.(\boldsymbol{\omega}\times\mathbf{r}) + (\boldsymbol{\omega}\times\mathbf{r})^2\right]$$

(b)**(4 marks)** Suppose that there are now numerous components in the system labelled by an index α. The total mass of the system is given by

$$M = \sum_\alpha m_\alpha$$

and the centre of mass is defined by,

$$\mathbf{r}_{cm} = \frac{1}{M}\sum_\alpha \mathbf{r}_\alpha m_\alpha.$$

Show that the total kinetic energy for this system is given by,

$$T = \frac{M}{2}{\mathbf{v}_b}^2 + M\mathbf{r}_{cm}.(\mathbf{v}_b\times\boldsymbol{\omega}) + \frac{1}{2}\sum_\alpha m_\alpha[r_\alpha^2\omega^2 - (\boldsymbol{\omega}.\mathbf{r}_\alpha)^2]$$

(c)**(2 marks)** We can simplify this equation still further by choosing to take the origin of the coordinate system at one specific place - where? What is the kinetic energy in this case?
(d)**(4 marks)** The inertia tensor for a continuous system can be written

$$I_{ij} \equiv \mathcal{I} = \int_{volume}\rho(\vec{r})d^3r\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2\end{pmatrix}$$

Suppose the inertia tensor is measured around the centre of mass, $\mathbf{r}_{cm}$. For a continuous system the centre of mass will be given by

$$\mathbf{r}_{cm} = \frac{1}{M}\int_{volume}\rho(\vec{r})\vec{r}d^3r$$

Show that if we instead measure around another point $\mathbf{r}' = \mathbf{r} - \mathbf{a}$ where $\mathbf{a}$ is a fixed displacement vector, (and $\mathbf{r}$ is measured in the centre of mass frame) then the revised inertia tensor is

$$\mathcal{I}' = \mathcal{I} + M\begin{pmatrix} a_y^2+a_z^2 & -a_xa_y & -a_xa_z \\ -a_ya_x & a_x^2+a_z^2 & -a_ya_z \\ -a_za_x & -a_za_y & a_x^2+a_y^2\end{pmatrix}$$

(2) Evaluation of inertia tensors. We derived the equation for the components of inertia tensor for a continuous system:

$$I_{jk} = \int_V \rho(\mathbf{r})(\mathbf{r}^2\delta_{jk} - r_jr_k)dxdydz$$

As an example let us calculate the inertia tensor for a cube, size 2 along each axis and mass m, (with constant density ρ) measured about its centre. The integral over the volume relates the mass to the density to give $8\rho = m$. Symmetry means that the off diagonal terms will be zero, and also the x, y, z diagonal terms will be equal. We need only calculate the xx (*i.e.* 11) term.

$$I_{11} = \int_V \rho(\mathbf{r})(\mathbf{r}^2\delta_{11} - xx)dxdydz = \rho\int_{-1}^1\int_{-1}^1\int_{-1}^1(y^2+z^2)dxdydz$$

$$I_{11} = 2\rho \int_{-1}^{1} \int_{-1}^{1} (y^2 + z^2) dy dz$$

$$I_{11} = 2\rho \int_{-1}^{1} \left[\frac{y^3}{3} + z^2 y \right]_{-1}^{1} dz = 2\rho \int_{-1}^{1} \left[\frac{2}{3} + 2z^2 \right] dz$$

$$I_{11} = 2\rho \left[\frac{2z}{3} + \frac{2z^3}{3} \right]_{-1}^{1} = \frac{16}{3}\rho = \frac{2}{3}m$$

(a)**(5 marks)** Show that the I_{33} component for a sphere of radius r and mass m, measured around the centre of the sphere is $2mr^2/5$. The integral over the volume is best done using spherical polar coordinates.
(b)**(3 marks)** Now assume the sphere is cut in half with the flat side placed against the x & y axes and the pole pointing along the z-axis. Show that the z-coordinate of the centre of mass is,

$$z_{cm} = \frac{3r}{8}.$$

(c)**(5 marks)** Re-centre the cut sphere so that the origin of the coordinate system is now at the centre of mass. Thus show that the I_{33} component for this cut sphere, as measured from the centre of mass, is $2mr^2/5$, where *m is the mass of the cut sphere.*

(3) Stability of rotation. Suppose we place a book face down on a surface. Rotation in this plane will have the maximum moment of inertia. The smallest moment of inertia will correspond to rotating the book through its centre of mass and around its long axis (the distance to edge of the book is smallest in this configuration). Rotation around the other direction will have an intermediate moment of inertia.
When you throw a book like a frisbee it tends to stay in that configuration. Similarly if you spin a book around its long axis that motion is reasonably stable. However, if you spin about the intermediate axis that motion is almost never stable. We can use Euler's equations to demonstrate why this is.
(a)**(5 marks)** Consider the case where I_1 is the largest moment of inertia ($I_1 > I_2 > I_3$, ω_1 is constant and then a small perturbation in the spin axis makes both ω_2 & ω_3 non-zero but still small. By differentiating the $I_2\dot{w}_2$ component in Euler's equations in the torque-free formulation and using appropriate substitutions, show that,

$$I_2\ddot{w}_2 - \frac{(I_1 - I_3)(I_2 - I_1)}{I_3}\omega_1^2\omega_2 = 0$$

(b)**(3 marks)** This equation is similar to simple harmonic motion:

$$\ddot{w}_2 + A\omega_2 = 0,$$

where

$$A = \frac{(I_1 - I_3)(I_1 - I_2)}{I_3 I_2}\omega_1^2.$$

What will be solution for $\omega_2(t)$? What does tell us about the physical behaviour of ω_2?
(c)**(3 marks)** Repeat the analysis for ω_3. What is the behaviour of w_3?
(d)**(3 marks)** Suppose we now orient the book to spin about the intermediate axis. In this case $I_2 > I_1 > I_3$. We will derive the same equations as before, but the value of A will have changed. Derive solutions for ω_2 and ω_3 in this case. What does this tell us about motion starting from this configuration?

(4) A string is attached to a ceiling while the other end is wrapped around a uniform cylinder of mass m, radius r and with a moment of inertia $I = mr^2/2$. The system is acted on by gravity, hence as the string unwinds the cylinder moves vertically downward.
(a)**(5 marks)** Derive the Lagrangian for this system.
(b)**(3 marks)** Hence show that if x is the vertical coordinate of the centre of mass then it must fall as

$$\ddot{x} = \frac{2g}{3}$$

(5) A disk of mass m, radius a, is attached to a bar of soap of mass m via a massless spring with spring constant k and a natural length of zero. The system is placed on a tilted surface with an angle β. Assume that as the system moves down the surface the soap moves without friction and the disk rotates without slipping. As the system moves the spring will lie between the two objects along a line connecting the centres of mass. Use a generalized coordinates for the distance each of the objects has moved down the surface.
(a)**(4 marks)** Derive the Lagrangian for the system.
(b)**(5 marks)** Show that if l is the distance the bar of soap has moved, and s the distance the disk has moved, the equations of motion are,

$$\ddot{l} + \frac{k}{m}(l - s) = g \sin \beta$$

$$\ddot{s} - \frac{2k}{3m}(l - s) = \frac{2}{3} g sin\beta$$

(c)**(3 marks)** Let's assume there is a solution in which the spring stays at a constant length. Calculate this length.