PHYS3300 - Assignment 3

Due in mailbox, Nov 3, 2023 (Time allowed=9 days)

Show all working to receive full credit, especially when performing integrals via substitution. If it is acceptable to quote a standard result then it will be mentioned in the question.

(1) (a)**(2 marks)** Compute the generalized momenta (see lecture 8) for the following free-body Lagrangian. Are the generalized momenta conserved? If so, what do they correspond to?

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

(b)**(4 marks)** Rewrite the free-body Lagrangian in spherical polar coordinates, including a central potential V(r). Derive the generalized momenta, state which are conserved and what they correspond physically to. Also derive the equations of motion.
(c)**(2 marks)** The equation of motion for r can be simplified. Rewrite it in terms of a conserved quantity you derived in part (b) and assume that a particle is initially at $\theta = \pi/2$ and that $\dot{\theta} = 0$.
Note these are quite straightforward examples of conserved quantities. The theory behind conserved quantities in Lagrangians is quite a bit more subtle than just looking at conserved momenta. It is really more about whether a particular change of coordinates preserves the form of the Lagrangian or not (in fact both forms of the free body Lagrangian preserve both angular and linear momentum but you wouldn't know that from looking at conserved momenta directly). Coordinate changes that preserve the form of the Lagrangian are called *symmetries*, and each symmetry of the Lagrangian has a corresponding conserved quantity. A theorem known as *Noether's Theorem* allows us to determine all the conserved quantities for a given Lagrangian.

(2) Suppose the surface of an ice sheet is ground to a shape such that a distance r from the center, the height of the surface (z) is given by $r^2 = az$. This surface is also oriented straight up, and there is a constant gravitational field acting in the z direction. A block of mass m can slide on the surface. In this question we'll evaluate the constraint Lagrange multiplier in terms one of the coordinates, from this you can find the constraint forces.
(a)**(1 mark)** What is the most natural coordinate system to use in this case?
(b)**(2 marks)** Given the block moves on the surface, write down the Lagrangian that defines its motion in the coordinate system.
(c)**(1 mark)** Sliding on the surface is a holonomic constraint, (say f), and it has an associated Lagrange multiplier λ. Give a formula for the constraint.
(d)**(3 marks)** Derive the equations of motion (there should be 3) with the constraint.
(e)**(2 marks)** Use derivatives of the constraint equation and one of the equations of motion to derive an equation in $\ddot{r}$ and other terms that include g somewhere.
(f)**(2 marks)** Next use another of the equations of motion to derive a formula that eliminates $\ddot{r}$.
(g)**(1 mark)** Using the simple definition of angular momentum in plane polar coords, substitute in your answer to (f) for $mr^2\dot{\theta} = J$ (which one of the equations of motion should tell you is a constant).
(h)**(3 marks)** This equation still has a term in $\dot{r}^2$ that we want to eliminate, we can do this by calculating the total energy $E = T + V$ and using the equation for that. Energy is conserved so let E be a constant. Substitute for z & $\dot{z}$ using the constraint and its derivatives, as well as any terms where you can substitute for J. You should be able to derive an equation for $\dot{r}^2$.
(i)**(3 marks)** Combine this with your answer to (g) to derive an expression for the Lagrange multiplier as a function of r and the constants in the problem.
(j)**(2 marks)** What are the forces (or generalized forces) associated with the constraint? (note you don't need to substitute for λ in the answer).

(3) Practice with the summation convention. In the following questions δ_{ij} is the Kronecker delta. Remember that summed indices can be relabelled with another index. Also note that when you have a sum of two terms *e.g.* $c_{ik} = a_{ik} + b_{ik}$ this does not imply any summation on i, k. However, $c_{ik}x_k = (a_{ik} + b_{ik})x_k$ would

imply a summation over k. Write each of the following expressions in a simpler form
(a)**(1 mark)** $y_k\delta_{ik}$
(b)**(1 mark)** $\delta_{lm}\delta_{ln}$
(c)**(1 mark)** $\delta_{ij}\delta_{jk}\delta_{kl}$
(d)**(1 mark)** $\delta_{ij}n_in_j$ (where n_i are components of a unit normal vector)
(e)**(1 mark)** $y_iy_j(a_{ij} - a_{ji})$
The alternating symbol ϵ_{ijk} in three dimensions is defined by,

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if (i,j,k) is (1,2,3),(2,3,1) or (3,1,2)} \\ -1 & \text{if (i,j,k) is (3,2,1),(1,3,2) or (2,1,3)} \\ 0 & \text{if i=j, or j=k, or k=i} \end{cases} \tag{1}$$

If a vector cross product is defined by $\mathbf{a} \times \mathbf{b}=\epsilon_{ijk}\hat{e}_ia_jb_k$ where $\hat{e}_i$ is the basis vector in the i direction (e.g. i=1 means x, i=2 means y etc), show *using the summation convention* that,
(f)**(2 marks)** $\mathbf{c}.(\mathbf{a} \times \mathbf{b}) = \epsilon_{ijk}c_ia_jb_k$ and that consequently $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = -\mathbf{b}.(\mathbf{a} \times \mathbf{c})$.
(g)**(3 marks)** Show using the identity

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$$

that with the summation convention,

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

and

$$\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn}.$$

Using the fact that $\partial x_i/\partial x_j = \delta_{ij}$ and $\nabla \equiv \hat{e}_i\partial/\partial x_i$ where $\hat{e}_i$ are basis vectors, prove the following relationship using the summation convention:
(h)**(2 marks)** $\nabla|\mathbf{x}| = \mathbf{x}/|\mathbf{x}|$ (HINT: think about what $\nabla(\mathbf{x}.\mathbf{x})$ looks like first.)

(4) I stated without proof in the lecture that for two matrices representing orthogonal transformations, A and B, then C=AB was also an orthogonal transformation.
(a)**(2 marks)** Write down an expression for elements of C, namely, C_{ij} using elements of A & B and the summation convention.
(b)**(1 mark)** I explicitly wrote out the orthogonality condition in the lectures, ($A^TA = I$, where A^T is the transpose of A and I is the identity matrix). Write down the orthogonality condition for the matrix C, but as a relation between matrix elements C_{ij}.
(c)**(2 marks)** Prove, using the summation convention, that the elements of C, $C_{ij} = A_{ik}B_{kj}$, obey the orthogonality condition.
(d)**(1 mark)** Prove this using matrix notation.
(e)**(2 marks)** Show that for a passive changes of coordinates, *i.e.* $(\mathbf{r})' = A\mathbf{r}$, that the vectors $(\mathbf{r})'$ and $\mathbf{r}$ are equal.