# PHYS3300 - Assignment 1

**Due in Class on Wed, Oct 4, 2023 (Time allowed=two weeks)**

**Show all working to receive full credit, especially when performing integrals via substitution. If it is acceptable to quote a standard result then it will be mentioned in the question.**

Notes: The assignments will challenge your technical skills and also your "translative skills" - how does a text-based based question turn into a mathematical problem that you need to solve? In many people's opinion this is what being a physicist (as opposed to being a mathematician) is about. Up to this point in your career many problems have been set in a step-by-step fashion to take you through things carefully. In real problem-solving situations you will need to figure out these steps for yourselves, but the good news is that you will develop this skill over time.

**(1)** Calculate the following line integrals
(a) **(2 marks)** $\int_C x^3 y\, ds$ where C is a line beginning at (0,0) and ending at (0,2).
(b) **(2 marks)** $\int_C \mathbf{v}.d\mathbf{r}$ where $\mathbf{v} = (xy, x-y)$ is a 2-d vector field and $d\mathbf{r} = (dx, dy)$, along the path C, where C follows a straight line from (0,0) to (0,2) and then from (0,2) to (3,2).

**(2)** (a) **(6 marks)** Workout the Euler-Lagrange equations for the following integrals

$$(i) \int y^2(1+y'^2)^{1/2}dx$$

$$(ii) \int y^2(1+y'^2)^2 dx$$

(b) **(2 marks)** The fictional Professor Brainard has invented a remarkable material called "flubber" that will stretch endlessly without feeling the effect of gravity. To show how remarkable the material is he takes two hoops and applies a ring of the flubber between them and then begins to move the two hoops apart. Supposing that one of the hoops is at positive $x = a$, and the other is it negative $x = -a$, write down the surface area of the flubber as an integral assuming it describes a circular tube of varying radius given by $y(x)$. (HINT: the surface area will depend on $ds$ and other factors.)
(c) **(5 marks)** The integral derived should depend on $y$ and $y'$ (where the prime denotes differentiation with respect to $x$). By writing it in a form,

$$S[y, y'] = \int_{-a}^{a} F(y, y')dx,$$

show that if the "flubber" takes the form of the minimum surface area, then finding the extremum for $S$ leads to,

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0.$$

(d) **(3 marks)** Hence show that $y(x)$ obeys the following differential equation:

$$\frac{1 + (y')^2 - y(y'')}{(1 + (y')^2)^{3/2}} = 0$$

(e) **(2 marks)** There is a way to solve this using an specific differential equation identity, but it is actually easier to use the second special case of the Euler-Lagrange equation we talked about in the lectures. Hence show that, if $c$ is a constant of integration,

$$y' = \sqrt{\frac{y^2}{c^2} - 1}$$

(f) **(2 marks)** By writing this in a form,

$$\int \frac{c}{(y^2 - c^2)^{1/2}} dy = \int dx$$

use the substitution $y = c \cosh u$ to complete the integrals and show that

$$y = c \cosh\left((x+\delta)/c\right),$$

is the solution for $y(x)$ where $\delta$ is another constant of integration.

**(3)** (a) **(4 marks)** In the notes we proved that the shortest path between two points in a Euclidean plane is the straight line. Derive the line element $ds$ on a *unit radius* cylinder using cylindrical polar coordinates $r, \theta, z$ (but think about what $r$ is) and then derive differential equations for the shortest path on a cylinder, firstly using the polar angle $\theta$ as the integration variable, and secondly using $z$ as the integration variable. Solve both equations. The Euler-Lagrange equation is the fastest way to derive the differential equations. (BONUS: what is the name of the curve describing the shortest path? The solution is also quite simple - can you think of a geometric reason for this?)
(b) **(4 marks)** There are a number of different ways of describing what's known as "hyperbolic geometries." The idea of different geometries is fundamental to Einstein's Theory of General Relativity. In this question we'll consider a simple example and what the shortest path (often called a geodesic) ends up looking like. Suppose the line element $ds^2$ in a geometry is given by,

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

Write the integral for the path length in terms of an integral over $y$ and $ds$ as we did for the shortest path example in class. From that use one of the special cases for the Euler- Lagrange equation to show that

$$\frac{dy}{dx} = \frac{1}{y}(c^2 - y^2)^{1/2}$$

where $c$ is a constant. Integrate this (HINT: you can substitute using a variable $v$ such that $y^2 = v$) and show that the resulting equation is $(x-d)^2 + y^2 = e$, where $e, d$ are constants, so that the shortest paths are actually circles!

**(4)** **(8 marks)** Most variational problems tend to depend upon a function $u$ and its derivative $u'$. An equation analogous to the Euler-Lagrange equation can be derived for problems involving the second derivative $u''$. Derive this equation by working through the variation principles outlined in the lecture notes. The only thing new in this question is the additional term in the expansion of the variation of the functional due to the dependence on $u''$. State any assumptions you make in the derivation. (NOTE: for physical problems where a system depends upon $\ddot{x}$ we must use this *generalized mechanics*, and these problems are called "jerky", jerk being the time derivative of the acceleration.)