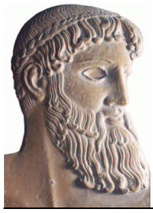


What is *planar* splitting?

Review of the *Method of Characteristics* (*MoC*; Stone & Norman, 1992):

- first successful attempt to upwind an MHD code (3 of 7 characteristics).
- Basic principle: *compressional* terms can be "operator split" from the *transverse* terms.

MHD equations:

$$\partial_t\rho + \nabla\cdot(\rho\,\vec{v}) = 0; \tag{1}$$

$$\partial_t e + \nabla\cdot(e\,\vec{v}) = -p\nabla\cdot\vec{v} \tag{2}$$

$$\partial_t\vec{v} + \vec{v}\cdot\nabla\vec{v} = \tfrac{1}{\rho}\left(-\nabla p^* + (\vec{B}\cdot\nabla)\vec{B}\right); \tag{3}$$

$$\partial_t\vec{B} + \nabla(\vec{B}\,\vec{v}) = \nabla(\vec{v}\,\vec{B}), \tag{4}$$

where $p = (\gamma - 1)e$ and $p^* = p + B^2/2$. $\vec{B}$ is in units where $\mu_0 = 1$.

Written out in components (with 3-symmetry assumed):

$$\partial_t\rho + \partial_1\rho v_1 + \partial_2\rho v_2 = 0; \tag{5}$$

$$\partial_t e + \partial_1 e v_1 + \partial_2 e v_2 = -p\nabla\cdot\vec{v}; \tag{6}$$

$$\partial_t v_1 + v_1\partial_1 v_1 + v_2\partial_2 v_1 = -\tfrac{1}{\rho}\partial_1 p^* + \tfrac{1}{\rho}B_1\partial_1 B_1 + \tfrac{1}{\rho}B_2\partial_2 B_1; \tag{7}$$

$$\partial_t v_2 + v_1\partial_1 v_2 + v_2\partial_2 v_2 = -\tfrac{1}{\rho}\partial_2 p^* + \tfrac{1}{\rho}B_1\partial_1 B_2 + \tfrac{1}{\rho}B_2\partial_2 B_2; \tag{8}$$

$$\partial_t B_1 + v_1\partial_1 B_1 + v_2\partial_2 B_1 = -B_1\nabla\cdot\vec{v} + B_1\partial_1 v_1 + B_2\partial_2 v_1; \tag{9}$$

$$\partial_t B_2 + v_1\partial_1 B_2 + v_2\partial_2 B_2 = -B_2\nabla\cdot\vec{v} + B_1\partial_1 v_2 + B_2\partial_2 v_2, \tag{10}$$

where blue terms are *compressive*, red terms are *transverse* (cross-derivatives).

Examining only the transverse terms for v_2 and B_2 [equations (8) and (10)]:

$$\partial_t v_2 + v_1\partial_1 v_2 = a_1\partial_1 a_2; \tag{11}$$

$$\partial_t a_2 + v_1\partial_1 a_2 = a_1\partial_1 v_2, \tag{12}$$

where $a_i = B_i/\sqrt{\rho}$ is the Alfvén speed. Adding and subtracting (11) and (12):

$$\partial_t v_2 + (v_1 - a_1)\,\partial_1 v_2 + \partial_t a_2 + (v_1 - a_1)\,\partial_1 a_2 \;=\; 0; \tag{13}$$

$$\partial_t v_2 + (v_1 + a_1)\,\partial_1 v_2 - \partial_t a_2 - (v_1 + a_1)\,\partial_1 a_2 \;=\; 0. \tag{14}$$

characteristic speeds: $c_i^{\pm} = v_i \mp a_i$;
Lagrangian derivatives: $D_{t,i}^{\pm} = \partial_t + c_i^{\pm}\partial_i \;\Rightarrow$

$$D_{t,1}^{+} v_2 + D_{t,1}^{+} a_2 = 0 \;\Rightarrow\; D_{t,1}^{+} c_2^{-} = 0; \tag{15}$$

$$D_{t,1}^{-} v_2 - D_{t,1}^{-} a_2 = 0 \;\Rightarrow\; D_{t,1}^{-} c_2^{+} = 0. \tag{16}$$

$\Rightarrow$ $c_2^{\mp}$ is the *Riemann invariant* along *characteristic* $\mathcal{C}_1^{\pm}$.
Similarly, $c_1^{\mp}$ is the Riemann invariant along characteristic $\mathcal{C}_2^{\pm}$.

Alternative form of the induction equation (4):

$$\partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{B}) \equiv \nabla \times \vec{E}, \tag{17}$$

where $\vec{E} = \vec{v} \times \vec{B}$ is the *induced electric field.* With 3-symmetry:

$$\partial_t B_1 = \partial_2 E_3; \quad \partial_t B_2 = -\partial_1 E_3, \tag{18}$$

where $E_3 = v_1 B_2 - v_2 B_1$.

Staggered mesh $\Rightarrow$	scalars (ρ, e)	zone-centred;
	primary vectors ($\vec{v}$, $\vec{B}$)	face-centred;
	secondary vectors ($\vec{E}$)	edge-centred.

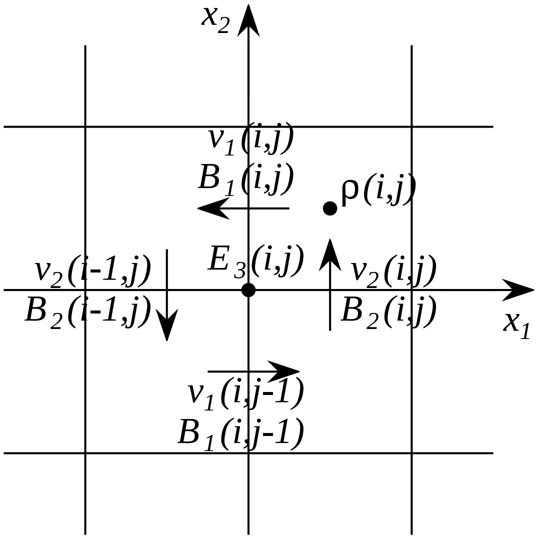
$\Rightarrow$ $\partial_2 E_3$ ($\partial_1 E_3$) co-spatial with B_1 (B_2), $\partial_t(\nabla \cdot \vec{B}) = 0$ to machine accuracy.

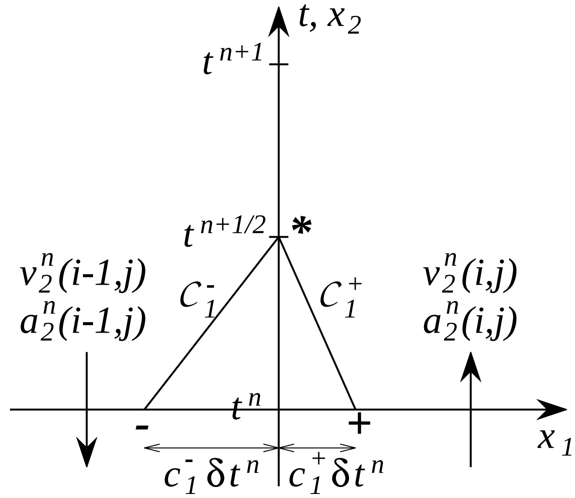
For E_3, need time-centred, upwinded interpolations of v_1, v_2, B_1, and B_2 to the 3-edges (v_1^*, v_2^*, B_1^*, and B_2^*).

For v_2^* and B_2^*, construct a space-time diagram centred on the 3-edge with the 1-characteristics arriving at the 3-edge at $t^{n+1/2}$ (t^*).

Difference (15) along $\mathcal{C}_1^{+}$ and (16) along $\mathcal{C}_1^{-}$ to get:

$$(v_2^* + a_2^*) - (v_{2,+} + a_{2,+}) = 0; \quad (v_2^* - a_2^*) - (v_{2,-} - a_{2,-}) = 0, \tag{19}$$





where $v_{2,\pm}$ and $a_{2,\pm}$ are v_2 and a_2 interpolated to the bases of $\mathcal{C}_1^\pm$ using the data at time t^n. Solve (19) for the "starred" quantities:

$$v_2^* = \tfrac{1}{2}\left(v_{2,+} + v_{2,-} + a_{2,+} - a_{2,-}\right); \quad a_2^* = \tfrac{1}{2}\left(a_{2,+} + a_{2,-} + v_{2,+} - v_{2,-}\right), \quad (20)$$

noting that $B_2^* = \sqrt{\rho^*}\, a_2^*$, with ρ^* some suitably centred value for ρ.

To locate the bases of the characteristics, one needs preliminary guesses for $c_1^\pm$ (call them $\tilde{c}_1^\pm$). The original MoC *used spatial averages:*

$$\tilde{c}_1^\pm = \tfrac{1}{2}\Big(v_1(i,j) + v_1(i,j-1)\Big) \mp \tfrac{1}{\sqrt{\rho^*}}\,\tfrac{1}{2}\Big(B_1(i,j) + B_1(i,j-1)\Big). \quad (21)$$

Repeat for v_1^* and B_1^*, then set:

$$E_3(i,j) = v_1^*(i,j)B_2^*(i,j) - v_2^*(i,j)B_1^*(i,j), \quad (22)$$

and use (18) to update B_1 and B_2.

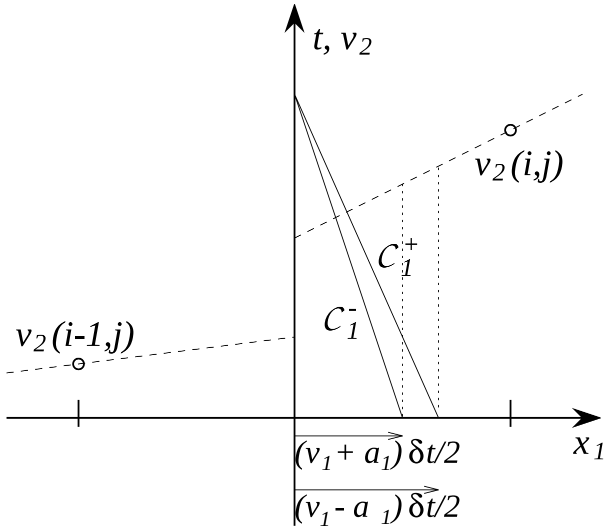
Difficulty with super-Alfvénic turbulence

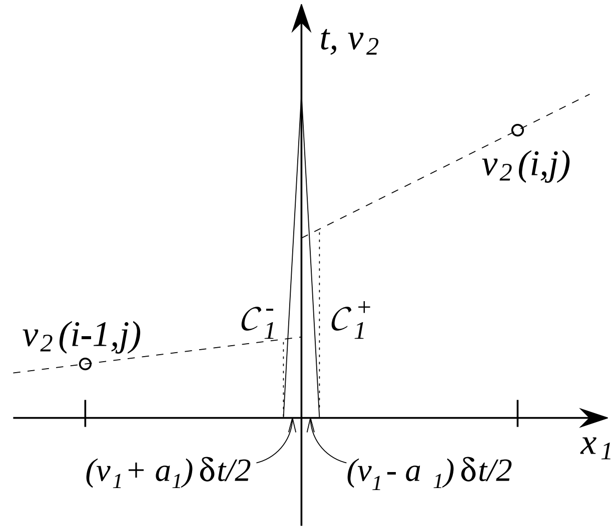
In general, $v_1 \gg a_1 \;\Rightarrow\; \mathcal{C}_1^\pm$ on same side of zone boundary $\;\Rightarrow\; v_{2,+} - v_{2,-} \propto a_1$, and a_2^* remains $\propto$ an *Alfvén* speed.

However, and on occasion (*e.g.*, shear layer),

$$v_1(i,j) \approx -v_1(i,j-1) \;\Rightarrow\; \tilde{v}_1(i,j) < \tilde{a}_1(i,j)$$

$\Rightarrow\; \tilde{c}_1^\pm$ have opposite sign $\;\Rightarrow\; \mathcal{C}_1^\pm$ *straddle* the zone boundary.



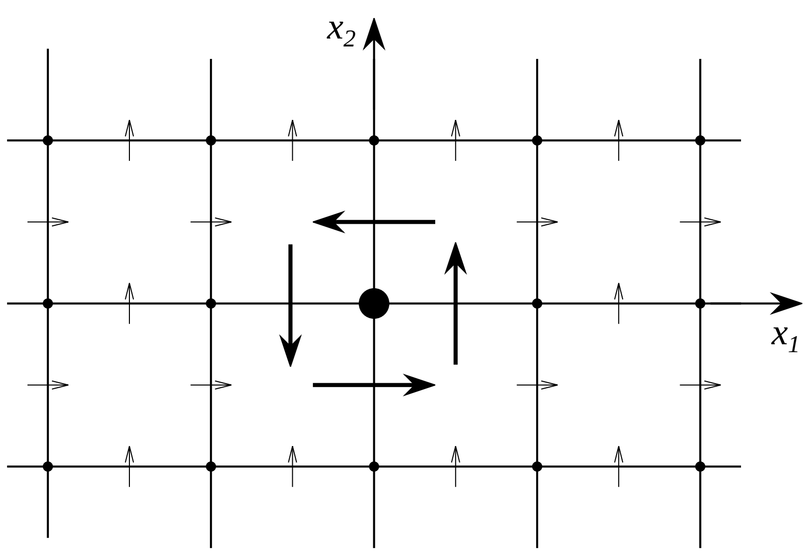


$\Rightarrow$ $v_{2,+} - v_{2,-}$ contains the jump between the left and right interpolation functions $\Rightarrow$ $a_2^* \sim v_2$, a *flow* speed.

This is not necessarily a bad thing! a_2^* (B_2^*) appears only in (22) when E_3 is evaluated. *If* $v_1^* \sim$ *an Alfvén speed when* $a_2^* \sim$ *a flow speed, the product* $v_1^* B_2^*$ *can remain physical.*

Indeed, it was $\tilde{v}_1 < \tilde{a}_1$ that caused $\mathcal{C}_1^{\pm}$ to straddle the zone boundary rendering $a_2^* \sim v_2$ in the first place. *But* $v_1^* \neq \tilde{v}_1$*!* Even if $\tilde{v}_1 < \tilde{a}_1$ (abnormal), $v_1^* \sim$ a flow speed (normal).

$\Rightarrow E_3 \sim v_1 v_2 \Rightarrow$ the magnetic flux loop around the errant induced electric field has an Alfvén speed $\sim$ a flow speed. This is a magnetic "explosion".



How frequent are explosions?

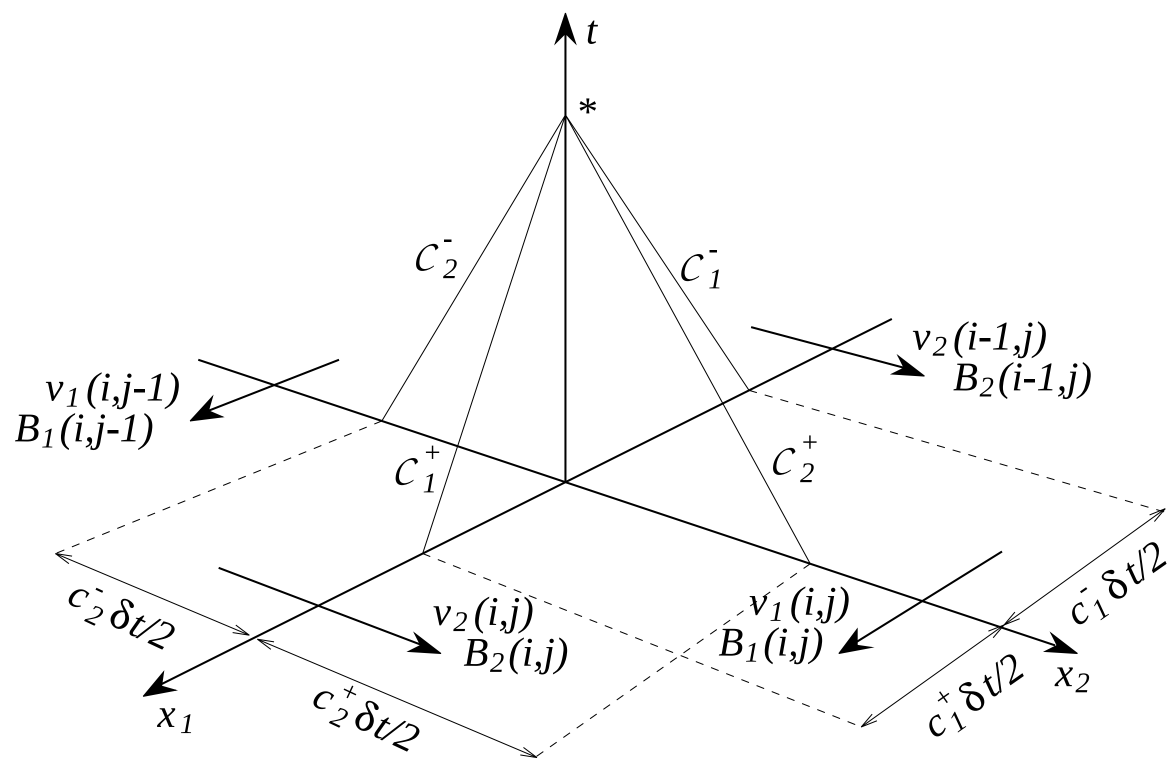
Let $a_1/v_1 \sim 10^{-6}$. $\tilde{v}_1 < \tilde{a}_1 \Rightarrow v_1(i,j) + v_1(i,j-1) < 10^{-6}$ which happens once every $\sim 10^6$ zone updates. For 10^4 time steps on a 128^3 grid $\Rightarrow$ 20,000 explosions, each injecting $\sim 10^6$ times the original magnetic energy into the grid!

Planar Splitting

Locate bases of $\mathcal{C}_1^\pm$ with v_1^* and a_1^* instead of $\tilde{v}_1$ and $\tilde{a}_1$. Then, if $v_1^* < a_1^*$ forces $\mathcal{C}_1^\pm$ to straddle the zone boundary and $a_2^* \sim v_2$, $\Rightarrow v_1^* \sim a_1 \Rightarrow E_3 \sim v_1^* a_2^*$ remains well behaved. *A "virtual exchange" of energies can be tolerated.*

Problem: v_1^* and a_1^* require bases of $\mathcal{C}_2^\pm$ to be set by v_2^* and a_2^*, and these can't be known until v_1^* and a_1^* are available to set the bases of $\mathcal{C}_1^\pm$!

This "catch-22" is solved by doing the interpolations *implicitly* on the 1-2 plane, identifying the "upwinded quadrant" instead of the traditional "upwinded direction". This is *planar-splitting* (*CMoC*, Clarke 1996).



Application to Godunov schemes

Directionally-split Godunov schemes [Ryu & Jones (1995), Dai & Woodward (1998), Falle *et al.* (1998)]:

- solve the seven-wave Riemann problem in each direction independently;
- account for discontinuities differently than MoC-based schemes and thus eliminate (?) "virtual exchanges" of kinetic and magnetic energies;
- set $\partial_i B_i = 0$, introducing directional biases (Gardiner & Stone, 2005);
- $\nabla \cdot \vec{B} = 0$ a problem.

Unsplit Godunov schemes [Balsara & Spicer (1999), Gardiner & Stone (2005), Ustyugov's "PPM-L"]:

- include the $\partial_i B_i$ terms;
- maintain $\nabla \cdot \vec{B} = 0$ to machine accuracy;
- susceptible to magnetic or kinetic explosions in sub-Alfvénic turbulence calculations (Dave Collins, *private communication*).

Additional interpolations to compute $\vec{E}$ from Riemann fluxes seem to make unsplit schemes vulnerable to virtual exchanges of kinetic and magnetic energies.