Lessons for

A first course in

Magnetohydrodynamics

Instructor's version

David Alan Clarke

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Part I 1-D MHD in Ten Weeks

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Preface to the instructor's version

This document contains the *instructor's version* of thirty-six lessons based on the textbook *A first course in Magnetohydrodynamics* (hereafter, "the text") suitable for a single-semester course with 24 – 36 classes. Each lesson is typically six pages of large-font double-spaced type designed for a 75-minute "flipped-style" class in which students are expected to have read the *student's version* of the lesson and any relevant portion of "the text" before coming to class. With this preparedness, the instructor can use class time to discuss the lesson notes and address students' questions rather than writing everything down on the board with students copying furiously. What the instructor elects to put on the board for clarity can be copied by students in between the lines of the lesson notes on a paper copy or in an editable PDF reader on their laptop. Each lesson concludes with a class exercise (in single-spaced, normal-sized font) designed to reinforce some of the main points of the lesson that can be done as part of a group discussion in 10 – 20 minutes.

The instructor's and student's versions of these lesson notes are not identical. Omitted from the student's version are:

1. answers to the end-of-class exercises; and
2. "demonstrations" such as web links for images and videos, embedded pictures with discussion, and suggested analogies for the instructor.

The intent is for the instructor to make available to the class at the beginning of term the entirety of the student's version of the lessons, whereas the instructor would teach from the instructor's version with the current lesson projected on the overhead. Portions omitted from their version would be seen by students for the first time during class. After each class, the instructor would replace the student's version of the lesson with the instructor's version so that students have access to the exercise answers and "demonstrations" after they've been presented and for further study. To do this, one merely opens up the LaTeX document `lessons_stu.latex`, scrolls down to the bottom, and replaces "`stu`" with "`ins`" for the lesson just taught. The document is then recompiled by typing at the UNIX prompt:

```
ml lessons_stu
```

where the script file "`ml`" that executes the necessary LaTeX commands is included in the same directory as `lessons_stu.latex` from where "`ml`" should be executed. This will update the PDF document `lessons_stu.pdf` which the instructor can upload to where students access the lesson notes. In this way, the student's version gradually becomes the instructor's version as the course unfolds.

Equations in the lesson notes are numbered by section, and thus have two decimals. For example, Eq. (3.1.4) is the fourth equation in §3.1 of Chap. 3. In "the text", equations are more coarsely numbered by chapter and have just one decimal; whence Eq. (3.24) is the 24^{th} equation in Chap. 3 without reference to the section. Thus, equation references in these notes with two decimals are to other equations in these notes, whereas equation references with just one decimal are to those in "the text". Further, while all chapter headings in the lesson notes follow those in "the text", beware that some of the section and subsection headings and numberings do not.

David Clarke, Halifax, NS.
October, 2024

Preface to the student's version

Welcome to your first course in magnetohydrodynamics (MHD), what I hope will be a memorable introduction to the fundamentals of how 99.99% of the baryonic universe operates!

This document contains thirty-six lessons based on the text, *A first course in Magnetohydrodynamics* (hereafter "the text"). Each lesson is typically six pages of large-font double-spaced type designed for a 75-minute "flipped-style" class in which you, the student, will be expected to have read (along with any relevant portion of "the text") before coming to class. With this preparedness, your instructor can use class time to discuss the lesson notes and address your questions rather than writing everything down on the board with you furiously copying before the board is erased! What your instructor elects to put on the board for clarity, you can copy down at a more leisurely pace in between the lines of a paper copy of these lesson notes or in an editable PDF reader on your laptop. Each lesson concludes with a class exercise (in single-spaced, normal-sized font) designed to reinforce some of the main points of the lesson and which your instructor may include as part of a group discussion.

While MHD is based on four very fundamental and familiar conservation laws (mass, energy, momentum, and magnetic flux), the mathematics is subtle and intricate, often masking the physical beauty behind its formality. It is therefore imperative for you, the student, to keep up with these notes and "the text" so that the "language of mathematics" doesn't "get in the way" but rather "speaks to you" as intended. Be sure to read each lesson before coming to class, and do not shy away from approaching your instructor even before class if there are key concepts you think you're missing.

On two practical matters, equations in the lesson notes are numbered by section, and thus have two decimals. For example, Eq. (3.1.4) is the fourth equation in §3.1 of Chap. 3. In "the text", equations are more coarsely numbered by chapter and have just one decimal; whence Eq. (3.24) is the 24^{th} equation in Chap. 3 without reference to the section. Thus, equation references in these notes with two decimals are to other equations in these notes, whereas equation references with just one decimal are to those in "the text". Further, while all chapter headings in the lesson notes follow those in "the text", beware that some of the section and subsection headings and numberings do not.

And with that, I wish you all a great adventure!

David Clarke, Halifax, NS.
October, 2024

Lesson 1

In this first lesson, areas of study in *continuum* and *particle dynamics* are defined, including,

1. *fluid* and *gas dynamics*;
2. *plasma physics*; and
3. *magnetohydrodynamics* (MHD).

The approach this course takes to study MHD is then identified.

The rest of the lesson is a review of the *Kinetic theory of gases*, where:

1. the applicable conservation laws are surveyed;
2. a link is made between *momentum transfer* and *pressure*;
3. a link is made between average particle *kinetic energy* and *temperature*;
4. the *rms speed* of gas particles at a given temperature is defined.

Introduction

Definition: A *fluid* is one of three states of matter that flows:

1. liquid – incompressible; ρ (density) = constant
2. gas – compressible
3. plasma – (perhaps partially) ionised gas

Physical property distinguishing each state: temperature.

<u>Formal definition of a fluid</u>

Let δl = mean free path of particles (distance between collisions)

Let $\mathcal{L}$ = any measurable scale of interest

- physically: smallest "turbule" as fluid cascades to turbulence
- numerically: one zone

A fluid is a medium in which $\delta l \ll \mathcal{L}$; "granularity" is sub-microscopic.

Fluid dynamics $\equiv$ *hydrodynamics* $\equiv$ *continuum mechanics*: physics of fluid flow, accounting for,

- conservation of mass
- conservation of energy
- Newton's second law

} (classical mechanics)

- an equation of state (thermodynamics)

Ideal fluid dynamics: assumes zero dissipation, *i.e.*, no viscosity (*inviscid*)

Gas dynamics: compressible fluid dynamics

Plasma physics: collective behaviour of e^- + ions with charge separation, $\lambda_\mathrm{D} \gtrsim \delta l$ ($\lambda_\mathrm{D} \equiv$ *Debye length*). The *Vlasov-Boltzmann equation* accounts for:

- collisions;
- electrodynamic forces arising from $\vec{E}$ and $\vec{B}$;
- conservation laws;
- Newtonian and/or relativistic dynamics,

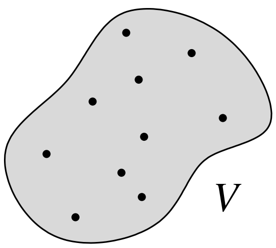
and is beyond scope of this course.

Magnetohydrodynamics (MHD) $\equiv$ plasma physics with $\lambda_\mathrm{D} \ll \delta l$, or fluid dynamics with non-zero $\vec{B}$. This course adopts latter approach.

- Fluid can't support "static" electric fields (free charges), but can support "induced" electric fields: $\dfrac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}_\mathrm{ind}$.
- Collective behaviour of e^- + ions $\Rightarrow$ $\vec{B}$ $\Rightarrow$ $\vec{J} = \nabla \times \vec{B}$ $\Rightarrow$ Lorentz force ($\vec{J} \times \vec{B}$).
- MHD = HD + Lorentz forces + induction equation.

Chapter 1. Fundamentals of Hydrodynamics

1.1 Kinetic Theory of Gases



Consider ensemble of $\mathcal{N}$ particles, mass m, in volume V.

"Walls" of V may be rigid or completely flexible so that they deform as needed to keep all $\mathcal{N}$ particles within V.

All particle-particle, particle-wall collisions are elastic.

Governing physical principles are known to first-year students:

$$\text{conservation of mass:} \quad \frac{dM}{dt} = 0; \qquad M = \mathcal{N}m$$

$$\text{conservation of energy:} \quad \frac{dE_{\mathrm{T}}}{dt} = \mathcal{P}_{\mathrm{app}}$$

$$\text{Newton's Second Law:} \quad \frac{d\vec{S}}{dt} = \sum \vec{F}_{\mathrm{ext}}; \qquad \vec{S} = m\sum_{i=1}^{\mathcal{N}} \vec{v}_i$$

where:

$\vec{F}_{\mathrm{ext}}$ = forces *external* to V acting on particles within V:

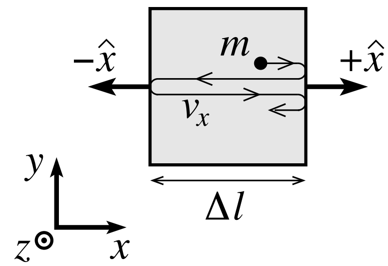
- collisions between particles outside V with "walls" of V ("applied")
- gravity
- magnetic
- viscous forces along "walls" of V

E_{T} = "total energy" = kinetic + internal + gravitational (+ magnetic)

$\mathcal{P}_{\mathrm{app}}$ = power (rate at which work is done) by *applied* forces

- external collisions only
- gravitational and magnetic energy already part of E_{T}

Big question: How do we account for collisions?



Consider single particle of mass m within cube of side Δl, velocity $\vec{v} = v_x \hat{x}$.

After collision with wall, $\vec{v}' = -v_x \hat{x}$

$\Rightarrow$ $\Delta \vec{S}_m = m\vec{v}' - m\vec{v} = -2mv_x\hat{x} =$ impulse to m.

Conserve momentum: $\Delta \vec{S}_x = 2mv_x\hat{x} =$ impulse delivered to right $(+\hat{x})$ wall.

At time $\Delta t = 2\Delta l / v_x$, particle again collides with right wall.

$\Rightarrow$ average rate of delivery of momentum to right wall, $\langle F_x \rangle$, is:

$$\langle F_x \rangle = \frac{\Delta S_x}{\Delta t} = \frac{2mv_x}{2\Delta l / v_x} = \frac{mv_x^2}{\Delta l}.$$

$\Rightarrow$ average "pressure" exerted by m against right wall is:

$$\langle p \rangle = \frac{\langle F_x \rangle}{\text{area}} = \frac{\langle F_x \rangle}{\Delta l^2} = \frac{mv_x^2}{\Delta l^3} = \frac{mv_x^2}{V}.$$

For $\mathcal{N}$ particles, all moving in $\hat{x}$-direction:

$$p = \sum_{i=1}^{\mathcal{N}} \langle p_i \rangle = \frac{m}{V} \sum_{i=1}^{\mathcal{N}} v_{x,i}^2 = \frac{\mathcal{N}m}{V} \underbrace{\frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} v_{x,i}^2}_{\langle v_x^2 \rangle} = \frac{\mathcal{N}m}{V} \langle v_x^2 \rangle, \tag{1.1.1}$$

where $\langle v_x^2 \rangle$ is the *mean square* of $v_{x,i}$ over the particle ensemble.

Now bring in fluid assumption: $\delta l \ll \Delta l \Rightarrow$ particle-particle collisions *isotropise* all motions,

$$\Rightarrow \quad \langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle \quad \text{(no direction preferred over others)}$$

$$\Rightarrow \quad \langle v^2 \rangle = \langle v_x^2 + v_y^2 + v_z^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle = 3\langle v_x^2 \rangle,$$

and Eq. (1.1.1) becomes:

$$p = \frac{\mathcal{N}m}{3V}\langle v^2\rangle = \frac{\mathcal{N}mv_{\rm rms}^2}{3V}, \tag{1.1.2}$$

where $v_{\rm rms} = \sqrt{\langle v^2\rangle}$ = *root-mean-square velocity.*

Now, from ideal gas law:

$$p = \frac{\mathcal{N}k_{\rm B}T}{V} \quad\Rightarrow\quad T = \frac{pV}{\mathcal{N}k_{\rm B}} = \frac{mv_{\rm rms}^2}{3k_{\rm B}} \quad \text{(substituting in Eq. 1.1.2)}$$

$$\Rightarrow \quad \frac{3}{2}k_{\rm B}T = \frac{1}{2}mv_{\rm rms}^2 = \langle K\rangle, \tag{1.1.3}$$

where $k_{\rm B} = 1.38\times10^{-23}\,{\rm J\,K^{-1}}$ (units of entropy) = Boltzmann constant,

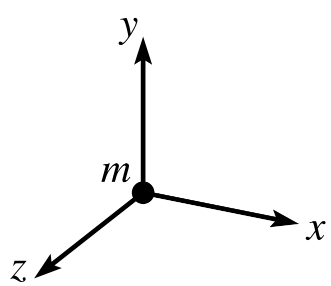
$\langle K\rangle$ = average kinetic energy per particle.

Interpretations:

- pressure (Eq. 1.1.2): rate of transfer of momentum from particles to surface (*e.g.*, wall, barometer diaphragm, skin, *etc.*);
- temperature (Eq. 1.1.3): proportional to mean particle kinetic energy.

Let E = total internal energy of all $\mathcal{N}$ particles (randomly directed K). Then,

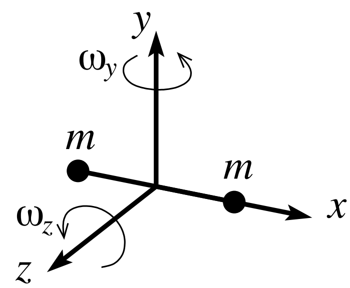
$$E = \mathcal{N}\langle K\rangle = \frac{3}{2}\mathcal{N}k_{\rm B}T = 3\underbrace{\left(\frac{1}{2}\mathcal{N}k_{\rm B}T\right)}_{E\text{ per “degree of freedom”}}$$



Point particle has three degrees of freedom: translational motion in each of x, y, z directions, each contributing $\frac{1}{2}\mathcal{N}k_{\rm B}T$ to internal energy.

Principle of Equipartition (PoE): Left to its own devices, a system distributes energy equitably among all degrees of freedom (DoF) available to it.

For a diatomic particle, $\exists$ three translational, two rotational, and two vibrational ($\frac{1}{2}kx^2, \frac{1}{2}mv^2$) DoF.



Stat Mech $\Rightarrow$ vibrational DoF dormant at ordinary temperatures, while PoE $\Rightarrow$ each rotational DoF contributes $\frac{1}{2}\mathcal{N}k_\mathrm{B}T$ of internal energy to ensemble.

$$\text{Thus, five DoF} \quad \Rightarrow \quad E_\text{diatomic} = \frac{5}{2}\mathcal{N}k_\mathrm{B}T.$$

Let ν = number of DoF, and define:

$$\gamma \equiv 1 + \frac{2}{\nu} \quad \Rightarrow \quad \nu = \frac{2}{\gamma - 1}$$

	ν	γ
monatomic	3	$\frac{5}{3}$
diatomic	5	$\frac{7}{5}$
polyatomic	$5 < \nu \leq 6$	$\frac{7}{5} < \gamma \leq \frac{4}{3}$,

where one can show $\gamma = C_P/C_V$ (ratio of specific heats), and where for an *adiabatic* gas of density ρ, $p \propto \rho^\gamma$.

For polyatomic molecules, Stat Mech and tensor moment of inertia $\Rightarrow$ $\nu \notin \mathbb{Z}$.

Still, difference between 7/5 and 4/3 for γ is generally unimportant.

Writing: $E = \dfrac{\nu}{2}\mathcal{N}k_\mathrm{B}T = \dfrac{\mathcal{N}k_\mathrm{B}T}{\gamma - 1}$, define *internal energy density* as,

$$e \equiv \frac{E}{V} = \frac{1}{\gamma - 1}\frac{\mathcal{N}k_\mathrm{B}T}{V} = \frac{p}{\gamma - 1} \quad \text{using ideal gas law}$$

$$\Rightarrow \quad \boxed{p = (\gamma - 1)e,} \tag{1.1.4}$$

most common form of ideal gas law in hydrodynamics. Another useful form:

$$p = \frac{\mathcal{N}mk_\mathrm{B}T}{mV} = \frac{M}{V}\frac{k_\mathrm{B}T}{m} \quad \Rightarrow \quad \boxed{p = \frac{\rho k_\mathrm{B}T}{m},} \tag{1.1.5}$$

where m is average gas particle mass.

Finally, revisiting our definition of $v_{\rm rms}$ (Eq. 1.1.3),

$$v_{\rm rms} = \sqrt{\frac{3k_{\rm B}T}{m}} = \sqrt{\frac{\nu k_{\rm B}T}{m}} = \sqrt{\frac{2k_{\rm B}T}{(\gamma-1)m}} = \sqrt{\frac{2p}{(\gamma-1)\rho}}. \qquad (1.1.6)$$

Preview: sound speed in an adiabatic gas is:

$$c_{\rm s} = \sqrt{\frac{\gamma p}{\rho}} \quad \Rightarrow \quad \frac{v_{\rm rms}}{c_{\rm s}} = \sqrt{\frac{2}{\gamma(\gamma-1)}} = \frac{3}{\sqrt{5}},$$

for $\gamma = 5/3$. That $v_{\rm rms}$ and $c_{\rm s}$ differ only by a constant of order unity suggests a fundamental relationship exists between them (Lesson 4).

Class exercises:

(Problem 1.1) On a cold winter afternoon, you enter your winter cabin (which has not been heated for weeks) freezing cold. You light a roaring fire in the hearth and after an hour, the cabin is warm enough to take off your winter clothing.

a) Does the air in your cabin contain more, less, or the same total internal energy, E, now that it is warm than when it was cold? Explain.

b) Where does all the energy from the fire go?

Answers.

a) From the ideal gas law (Eq. 1.1.4), $e = p/(\gamma-1)$ and thus the total internal energy in the cabin of volume V is:

$$E = eV = \frac{pV}{\gamma-1}.$$

Evidently, V is constant and, despite the warmer temperature, the cabin pressure, p, also remains constant (otherwise the door could burst open!). Thus E is unchanged.

So what does change? From Eq. (1.1.5), $p \propto \rho T$. For T to increase while p remains constant means ρ must decrease; hot air, being less dense, rises.

b) Some energy from the fire heats up the solids in the room—the furniture, walls, *etc.* A substantial amount of energy does work to force air through the walls, leaky doors, and windows where the warmer cabin air both heats and expands into the colder air outdoors. In Lesson 2 we shall refer to this expansion as "*pdV work*".

Lesson 2

In this lesson, we introduce the *Theorem of hydrodynamics*, from which the three fundamental equations of ideal hydrodynamics are derived:

1. the *continuity equation*;
2. the *total energy equation*; and
3. the *momentum equation*.

1.2 Theorem of hydrodynamics

Definitions

Extensive quantity: proportional to amount of material (*e.g.*, M, E, V, *etc.*, represented in upper case)

Intensive quantity: independent of amount of material (*e.g.*, ρ, e, p, T, *etc.*, represented in lower case; T an exception)

$\forall$ extensive quantity, Q, $\exists$ a corresponding intensive quantity, q, such that:

$$q(\vec{r},t) \;=\; \lim_{\Delta V \to 0} \frac{\Delta Q(V,t)}{\Delta V} \;=\; \frac{\partial Q(V,t)}{\partial V}; \qquad Q(V,t) \;=\; \int_V q(\vec{r},t)\, dV.$$

- $q(\vec{r},t)$ must be *integrable* (no poles of $\geq 1^{\text{st}}$ order) over V
- $Q(V,t)$ must be *differentiable* over V (no discontinuities or any order poles); more restrictive than integrability.

Theorem 1.1. *Theorem of hydrodynamics. If $Q(t)$ is an extensive quantity with $q(\vec{r},t)$ its corresponding intensive quantity, then:*

$$\frac{dQ}{dt} = \Sigma \quad \Longleftrightarrow \quad \frac{\partial q}{\partial t} + \nabla \cdot (q\vec{v}) \;=\; \sigma,$$

where $\vec{v} = d\vec{r}/dt$, and $\Sigma = \int_V \sigma\, dV$ are "source terms". Note that the product $q\vec{v}$ must be differentiable *over V.*

Proof:
$$\frac{dQ}{dt} = \Sigma \quad \Longleftrightarrow \quad \frac{d}{dt}\int_V q dV \;=\; \int_V \sigma dV,$$

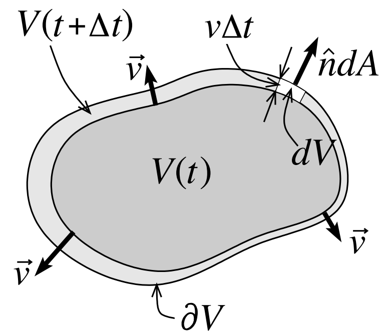
where $V = V(t)$ may also vary in time. Thus,

$$\frac{d}{dt}\int_{V(t)} q dV \;=\; \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} q(\vec{r},t+\Delta t) dV - \int_{V(t)} q(\vec{r},t) dV\right]$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V(t+\Delta t)-V(t)} q(\vec{r},t+\Delta t)dV + \int_{V(t)} q(\vec{r},t+\Delta t)dV - \int_{V(t)} q(\vec{r},t)dV\right]$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\Delta V} q(\vec{r},t+\Delta t)dV + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{V(t)}\big[q(\vec{r},t+\Delta t) - q(\vec{r},t)\big]dV.$$

Integrating over $\Delta V = V(t+\Delta t) - V(t)$ is same as integrating over closed surface, ∂V, with volume differential $dV = (\vec{v}\Delta t)\cdot(\hat{n}dA)$. Thus,



$$\frac{d}{dt}\int_{V(t)} q dV = \lim_{\Delta t\to 0}\frac{1}{\cancel{\Delta t}}\oint_{\partial V} q(\vec{r},t+\Delta t)(\vec{v}\cancel{\Delta t})\cdot(\hat{n}dA) + \int_{V(t)}\lim_{\Delta t\to 0}\frac{q(\vec{r},t+\Delta t) - q(\vec{r},t)}{\Delta t}dV$$

$$= \oint_{\partial V} q(\vec{r},t)\vec{v}\cdot\hat{n}dA + \int_{V(t)}\frac{\partial q(\vec{r},t)}{\partial t}[1] dV$$

$$= \int_{V(t)}\nabla\cdot\big(q(\vec{r},t)\vec{v}\big)dV + \int_{V(t)}\frac{\partial q(\vec{r},t)}{\partial t}dV \quad \text{(Gauss' theorem)}$$

$$= \int_{V(t)}\left(\frac{\partial q(\vec{r},t)}{\partial t} + \nabla\cdot\big(q(\vec{r},t)\vec{v}\big)\right)dV = \int_{V(t)}\sigma(\vec{r},t)dV.$$

$$\Rightarrow \quad \int_V\left(\frac{\partial q}{\partial t} + \nabla\cdot(q\vec{v}) - \sigma\right)dV = 0,$$

true for any V. Thus, integrand must be zero, proving the theorem. □

- Q is the conserved quantity (modulo Σ); q is *volume-conservative.*
- $\vec{f}_Q \equiv q\vec{v}$ is the *advective Q-flux density* (units $[Q]\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$).
- $\mathcal{F}_Q \equiv \int_\Sigma \vec{f}_Q\cdot\hat{n}\,dA =$ *advective Q-flux* (units $[Q]\,\mathrm{s}^{-1}$)
  $=$ *flux* of $\vec{f}_Q$ through surface Σ (units $[\vec{f}_Q]\,\mathrm{m}^2$).

[1] *Note to instructor*: Students may need reminding why it's the partial derivative here, and not the full.

- $\vec{f}_Q$ also interpreted as the *flux density* of $\mathcal{F}_Q$ (units $[\mathcal{F}_Q]\,\mathrm{m}^{-2}$).

1.3 Conservative equations of ideal HD

Definition: *Ideal HD* means no dissipation of any sort.

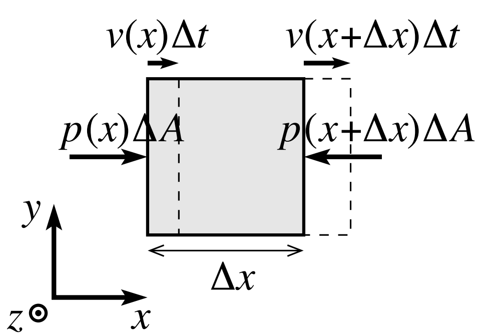
- no viscosity (*inviscid*), radiation, resistivity (MHD), *etc.*

1. Continuity: Let $Q = M$; $q = \rho$. Conservation of mass $\Rightarrow \dfrac{dM}{dt} = 0$.

Thus, HD theorem $\Rightarrow$
$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \;=\; 0,} \tag{1.3.1}$$
where $\rho\vec{v}$ is the advective mass flux density.

2. Total energy equation: Let $Q = E_\mathrm{T}$; $q = e_\mathrm{T} = \frac{1}{2}\rho v^2 + e + \rho\phi$.

Conservation of energy $\Rightarrow \dfrac{dE_\mathrm{T}}{dt} = \sum \mathcal{P}_\mathrm{app}$. Thus, HD theorem $\Rightarrow$
$$\frac{\partial e_\mathrm{T}}{\partial t} + \nabla \cdot (e_\mathrm{T}\vec{v}) \;=\; \sum p_\mathrm{app}, \tag{1.3.2}$$
where p_app is the *applied power density* ($\mathrm{J\,s^{-1}m^{-3}}$).



Consider small "cube" of fluid, mass ΔM, volume ΔV, side Δx, cross sectional area $\Delta A = \Delta V/\Delta x$. Work done on left face is:
$$F_x(x)v_x(x)\Delta t \;=\; p(x)\underbrace{\Delta A\, v_x(x)\Delta t}_{\delta V(x)},$$
while on right face:
$$F_x(x{+}\Delta x)v_x(x{+}\Delta x)\Delta t \;=\; \underset{\uparrow}{-}p(x{+}\Delta x)\underbrace{\Delta A\, v_x(x{+}\Delta x)\Delta t}_{\delta V(x+\Delta x)}$$

$F_x(x{+}\Delta x)$ and $v_x(x + \Delta x)$ anti-parallel

$$\begin{aligned}
\Rightarrow \quad \Delta W &= p(x)\delta V(x) - p(x+\Delta x)\delta V(x+\Delta x) \quad (\text{“}pdV\text{” work}) \\
&= -\frac{p(x+\Delta x)\, v_x(x+\Delta x) - p(x)\, v_x(x)}{\Delta x} \underbrace{\Delta x \Delta A}_{\Delta V} \Delta t \\
&= -\frac{\partial (p v_x)}{\partial x} \Delta V \Delta t \\
\Rightarrow \quad \frac{\Delta W}{\Delta t} &= \mathcal{P}_{\rm app} = -\frac{\partial (p v_x)}{\partial x} \Delta V, \\
\Rightarrow \quad p_{\rm app} &= \frac{\mathcal{P}_{\rm app}}{\Delta V} = -\frac{\partial (p v_x)}{\partial x} \quad (x\text{-contribution only})
\end{aligned}$$

Including y- and z-contributions, $\sum p_{\rm app} = -\nabla \cdot (p\vec{v})$, and Eq. (1.3.2) $\Rightarrow$

$$\frac{\partial e_{\rm T}}{\partial t} + \nabla \cdot (e_{\rm T}\vec{v}) = -\nabla \cdot (p\vec{v}) \quad \Rightarrow \quad \boxed{\frac{\partial e_{\rm T}}{\partial t} + \nabla \cdot \big((e_{\rm T} + p)\,\vec{v}\big) = 0,} \tag{1.3.3}$$

the *total energy equation* where $(e_{\rm T} + p)\,\vec{v}$ = advective energy flux density.

3. Momentum equation: Let $Q = \vec{S}$; $q = \vec{s} = \rho\vec{v}$.

Newton's 2nd Law $\Rightarrow \dfrac{d\vec{S}}{dt} = \sum \vec{F}_{\rm ext}$. Thus, HD theorem $\Rightarrow$

$$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s}\vec{v}) = \sum \vec{f}_{\rm ext}, \tag{1.3.4}$$

where $\vec{s}\vec{v}$ is the *dyadic* (outer) product of $\vec{s}$ and $\vec{v}$:

$$\vec{s}\vec{v} = |s\rangle\langle v|^{2} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \begin{bmatrix} v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} s_x v_x & s_x v_y & s_x v_z \\ s_y v_x & s_y v_y & s_y v_z \\ s_z v_x & s_z v_y & s_z v_z \end{bmatrix}.$$

Thus[3],

$$\nabla \cdot (\vec{s}\vec{v}) = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} s_x v_x & s_x v_y & s_x v_z \\ s_y v_x & s_y v_y & s_y v_z \\ s_z v_x & s_z v_y & s_z v_z \end{bmatrix}$$

[2] "Bra-ket" notation due to Dirac; see footnote 5 on page 30 of the text.

[3] valid in Cartesian coordinates only; see App. A in text for other coordinate systems.

$$= \left(\frac{\partial(s_x v_x)}{\partial x} + \frac{\partial(s_x v_y)}{\partial y} + \frac{\partial(s_x v_z)}{\partial z}, \frac{\partial(s_y v_x)}{\partial x} + \frac{\partial(s_y v_y)}{\partial y} + \frac{\partial(s_y v_z)}{\partial z}, \frac{\partial(s_z v_x)}{\partial x} + \frac{\partial(s_z v_y)}{\partial y} + \frac{\partial(s_z v_z)}{\partial z} \right)$$

$$= \big(\nabla \cdot (s_x \vec{v}), \nabla \cdot (s_y \vec{v}), \nabla \cdot (s_z \vec{v})\big).$$

Next, $\vec{f}_{\rm ext}$ are external force densities ($\mathrm{N\,m^{-3}}$):

- pressure gradient;
- gravity (if any).

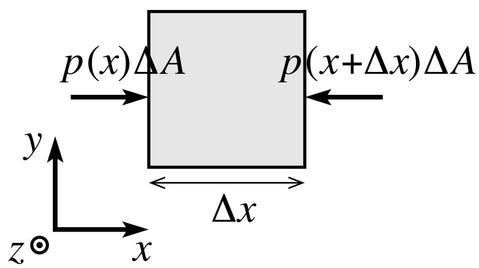
Net pressure force in x-direction for small cube of fluid is:

$$\sum F_x = F(x+\Delta x) + F(x) = -p(x+\Delta x)\,\Delta A + p(x)\,\Delta A$$

$$= -\frac{p(x+\Delta x) - p(x)}{\Delta x} \underbrace{\Delta x\,\Delta A}_{\Delta V} = -\frac{\Delta p}{\Delta x}\Delta V \to -\frac{\partial p}{\partial x}\Delta V$$

$$\Rightarrow \quad \sum f_x = \frac{1}{\Delta V}\sum F_x = -\frac{\partial p}{\partial x}$$

$$\Rightarrow \quad \vec{f}_p = -\frac{\partial p}{\partial x}\hat{x} - \frac{\partial p}{\partial y}\hat{y} - \frac{\partial p}{\partial z}\hat{z} = -\nabla p.$$



For gravity, $\vec{F}_g = -\Delta M\,\nabla\phi$

$$\Rightarrow \quad \vec{f}_g = -\frac{\Delta M}{\Delta V}\nabla\phi = -\rho\nabla\phi.$$

Including $\vec{f}_p$, $\vec{f}_g$ on RHS of Eq. (1.3.4) yields the *momentum equation*:

$$\boxed{\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s}\vec{v}) = -\nabla p - \rho\nabla\phi.} \tag{1.3.5}$$

Using identity $\nabla p = \nabla \cdot (p\mathbb{I})$ where $\mathbb{I}$ = identity matrix (exercises), we get,

$$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s}\vec{v} + p\mathbb{I}) = -\rho\nabla\phi, \tag{1.3.6}$$

where $\vec{s}\vec{v} + p\mathbb{I}$ = advective momentum flux density.

Class exercises:

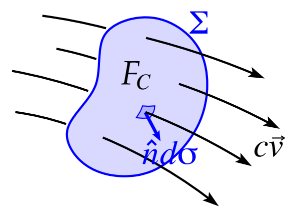
1. Let C be an extensive (hydrodynamical) quantity and c its corresponding intensive quantity. Suppose C is conserved, and thus,

$$\frac{dC}{dt} = 0.$$

   a) Write down the evolution equation for the intensive quantity, c.

   b) In terms of C, what kind of flux is $c\vec{v}$? Justify by noting the units.

   c) In terms of C, what kind of flux is $\mathcal{F}_C = \displaystyle\int_\Sigma (c\vec{v}) \cdot \hat{n} d\sigma$, where Σ is an open surface as illustrated in the inset.



   d) In terms of $\mathcal{F}_C$, what kind of flux is $c\vec{v}$?

   e) In terms of $c\vec{v}$, what kind of flux is $\mathcal{F}_C$?

2. Show that for *any* coordinate system, $\nabla p = \nabla \cdot (p\mathbb{I})$, where $\mathbb{I}$ is the identity matrix.

Answers.

1. *a*) $\dfrac{\partial c}{\partial t} + \nabla \cdot (c\vec{v}) \;=\; 0$ (Theorem of hydrodynamics).

   b) Units of $c\vec{v}$ are $([C]\mathrm{m}^{-3})(\mathrm{m\,s}^{-1}) = [C]\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$, thus $c\vec{v} =$ *advective flux density of* C.

   "density" ↗ (m^{-2}) ↖ "advective" (s^{-1})

   c) Units of $\mathcal{F}_C$ are $[c\vec{v}]\mathrm{m}^2 = [C]\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}\,\mathrm{m}^2 = [C]\,\mathrm{s}^{-1}$, thus $\mathcal{F}_C =$ *advective flux of* C.

   d) Units of $c\vec{v}$ are $[\mathcal{F}_C]\,\mathrm{m}^{-2}$, thus $c\vec{v} =$ *flux density of* $\mathcal{F}_C$.

   e) Units of $\mathcal{F}_C$ are $[c\vec{v}]\,\mathrm{m}^2$, thus $\mathcal{F}_C =$ *flux of* $c\vec{v}$.

2. One cannot approach a problem like this by breaking up the vectors and matrices into their Cartesian components, for then one only shows the identity true for Cartesian coordinates. To prove true for *any* coordinate system, one *must* use vector identities such as those in App. A in the text. Thus, from Eq. (A.24),

$$\nabla \cdot (p\mathbb{I}) \;=\; \mathbb{I} \cdot \nabla p + p \nabla \cdot \mathbb{I} \;=\; \nabla p,$$

as desired, since $\mathbb{I}$ is the identity (and constant) matrix.