#### A First Course in Magnetohydrodynamics

This text introduces readers to magnetohydrodynamics (MHD), the physics of ionised fluids. Traditionally, MHD is taught as part of a graduate curriculum in plasma physics. By contrast, this text – one of a very few – teaches MHD exclusively from a fluid dynamics perspective, making it uniquely accessible to senior undergraduate students. Part I of the text uses the MHD Riemann problem as a focus to introduce the fundamentals of MHD: Alfvéns theorem, waves, shocks, rarefaction fans, and so on. Part II builds upon this with presentations of broader areas of MHD: fluid instabilities, viscid hydrodynamics, steady-state MHD, and non-ideal MHD. Throughout the text, more than 125 problems and several projects (with solutions available to instructors) reinforce the main ideas. In addition, largefont lesson plans for a "flipped-style" class are available free of charge to instructors who use this text as required reading for their course. This book is suitable for advanced undergraduate and beginning graduate students of physics, requiring no previous knowledge of fluid dynamics or plasma physics.

**David Clarke** is a retired professor of astronomy and physics from Saint Mary's University in Halifax, Nova Scotia. Over his thirty-year career, he has taught numerous courses in physics and astronomy at the undergraduate and graduate levels, including courses in fluid dynamics and MHD that inspired this text. He is co-developer of the original ZEUS MHD code and currently the primary developer of ZEUS-3D that he uses for his research in astrophysical jets and has made available open source to hundreds of investigators worldwide.



Professor Hannes Olof Gösta Alfvén (1908–1995), father of magnetohydrodynamics, 1970 Nobel Prize laureate for physics. Portrait created in 1972 by Benno Movin-Hermes (1902– 1977) using a *three-colour foil method* developed by the artist. Reproduced with permission from I. Movin and the Moderna Museet, Stockholm.

## A First Course in Magnetohydrodynamics

David Alan Clarke Saint Mary's University, Halifax NS Canada





Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment, a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org Information on this title: www.cambridge.org/9781009381475

DOI: 10.1017/9781009381468

© David Alan Clarke 2025

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press & Assessment.

When citing this work, please include a reference to the DOI 10.1017/9781009381468

First published 2025

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication Data Names: Clarke, David Alan, 1958– author. Title: A First Course in Magnetohydrodynamics / David Alan Clarke, Saint Mary's University, Nova Scotia. Description: Cambridge, United Kingdom ; New York, NY : Cambridge University Press, 2025. | Includes bibliographical references and index. Identifiers: LCCN 2024014429 | ISBN 9781009381475 (hardback) | ISBN 9781009381468 (ebook) Subjects: LCSH: Magnetohydrodynamics – Textbooks. Classification: LCC QC718.5.M36 C53 2025 | DDC 538/.6-dc23/eng/20240629 LC record available at https://lccn.loc.gov/2024014429 ISBN 978-1-009-38147-5 Hardback

Cambridge University Press & Assessment has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate. For Jodi, my life-long love.

# Contents

Ρ	reface	2	xiii
In	Introduction		1
Ρ	art I	1-D MHD in Ten Weeks	5
1	The	Fundamentals of Hydrodynamics	7
	1.1	Definition of a fluid	7
	1.2	A quick review of kinetic theory	9
	1.3	The equations of ideal hydrodynamics	12
	1.4	The internal energy density	18
	1.5	Primitive, integral, and conservative form	20
	Prot	olem Set 1	22
2	Sele	cted Applications of Hydrodynamics	26
	2.1	Sound waves	26
		2.1.1 Wave equation approach	26
		2.1.2 Eigenvalue approach	29
	2.2	Rankine–Hugoniot jump conditions	33
		2.2.1 Case 1: Trivial solution	35
		2.2.2 Case 2: Tangential discontinuity	35
		2.2.3 Case 3: Shock	36
	2.3	Bores and hydraulic jumps (optional)	42
		2.3.1 Bores in the "lab frame"	46
	2.4	Bernoulli's theorem	47
	Prob	blem Set 2	56
3	The	Hydrodynamical Riemann Problem	69
Ĩ	3.1	Eulerian and Lagrangian frames of reference	70
	3.2	The three characteristics of hydrodynamics	72
	3.3	Characteristic paths and space-time diagrams	74
	3.4	The MoC and the Riemann problem	78
	3.5	Non-linear hydrodynamical waves	81
	0.0	3.5.1 Hyperbolic system of equations	81

		3.5.2 Left and right eigenvectors (optional) $\ldots \ldots \ldots \ldots$	82
		3.5.3 Hydrodynamical rarefaction fans	85
	3.6	Solution to the HD Riemann problem	90
	Prob	lem Set 3	94
4	The	Fundamentals of Magnetohydrodynamics	99
	4.1	A brief introduction to MHD	99
	4.2	The ideal induction equation	101
	4.3	Alfvén's theorem	103
	4.4	Modifications to the momentum equation	106
	4.5	The MHD Poynting power density	107
	4.6	Modifications to the total energy equation	108
	4.7	The equations of ideal MHD	109
	4.8	Vector potential and magnetic helicity (optional)	112
		4.8.1 Magnetic topology	114
	Prob	$\operatorname{lem}\operatorname{Set} 4 \ldots $	117
5	MHI	D Waves and Discontinuities	123
	5.1	Primitive and conservative equations of MHD	124
	5.2	MHD wave families	127
		5.2.1 (Shear) Alfvén waves	130
		5.2.2 Fast and slow magnetosonic waves	139
		5.2.3 Summary of MHD waves	146
	5.3	The MHD Rankine–Hugoniot jump conditions	149
		5.3.1 Case 1: Trivial solution	154
		5.3.2 Case 2: Contact discontinuity	154
		5.3.3 Case 3: Tangential discontinuity	154
		5.3.4 Case 4: Rotational discontinuity $\ldots \ldots \ldots \ldots \ldots \ldots$	155
		5.3.5 Case 5: Fast, slow, and intermediate shocks	155
	Prob	lem Set 5	172
6	The	MHD Riemann Problem	183
	6.1	Overview	183
	6.2	Non-linear MHD waves	185
		6.2.1 Fast and slow eigenkets	186
		6.2.2 Fast and slow rarefaction fans	192
	6.3	Space-time diagrams	202
	6.4	An MHD Riemann solver	204
		6.4.1 Problem parameters	205
		6.4.2 Strategy for the Riemann solver	208
		6.4.3 Algorithm for an exact MHD Riemann solver	210
		6.4.4 Sample problems	221
	Prob	lem Set 6	235

#### Part II Additional Topics in (M)HD

າ	Λ	1
4	+	т

7	Fluid	Instat	bilities	243	
•	7.1 Kolvin-Holmholtz instability				
	1.1	711	Normal mode analysis of the KHI	240	
		712	The development of the KHI	251	
		7.1.3	The KHI in nature	252	
		7.1.4	Numerical analysis of the KHI (optional)	254	
	7.2	Ravlei	igh-Taylor instability	256	
		7.2.1	Normal mode analysis of the RTI	257	
		7.2.2	Numerical analysis of the RTI (optional)	262	
		7.2.3	Kruskal–Schwarzchild instability	267	
	7.3	Magne	eto-rotational instability	268	
		7.3.1	Mathematical model of the MRI	269	
		7.3.2	Physical model of the MRI	275	
		7.3.3	Angular momentum transport	278	
		7.3.4	Numerical analysis of the MRI (optional)	283	
	7.4	Parker	r instability	286	
		7.4.1	A qualitative description	286	
		7.4.2	A quantitative description (optional)	291	
	Prob	lem Set	$\mathbf{t}$ $7$ $.$	305	
8	Visci	d Hydı	rodynamics	309	
	8.1	Introd	luction	309	
	8.2	The st	tress tensor	310	
		8.2.1	The trace of the stress tensor	312	
		8.2.2	Viscosity and Newtonian fluids	314	
		8.2.3	Non-isotropic "pressure"	315	
	8.3	The N	lavier–Stokes equation	317	
	8.4	The v	iscid energy equation	320	
		8.4.1	Viscous dissipation	321	
	8.5	The R	teynolds number	322	
	8.6	Applie	cations	326	
		8.6.1	Plane laminar viscous flow	327	
		8.6.2	Forced flow between parallel plates	328	
		8.6.3	Open channel flow	329	
		8.6.4	Hagen–Poiseuille flow	333	
		8.6.5	Couette flow	334	
	Prob	lem Set	t 8	337	
q	Stea	dv-stat	te MHD	341	
-	9.1	The V	Veber–Davis constants	342	
	9.2	The N	IHD Bernoulli function	349	
		9.2.1	Critical launching angle	354	

9.3	Stellar winds	357
	9.3.1 Critical points	359
	9.3.2 The Weber–Davis Model	361
9.4	Astrophysical jets (optional)	366
Prob	blem Set 9	371
10 Non	-ideal MHD	378
10.1	Introducing non-ideal MHD	378
10.2	The three players	380
	10.2.1 A weakly ionised, isothermal, one-fluid model	380
	10.2.2 Relative importance of the non-ideal terms	387
10.3	Resistive dissipation	391
	10.3.1 The resistive induction equation	391
	10.3.2 Dissipation of magnetic energy	392
	10.3.3 Magnetic diffusion and reconnection	393
	10.3.4 Dynamo theory (optional)	399
10.4	The Hall effect	406
	10.4.1 The case of a completely ionised fluid	408
	10.4.2 Magnetic reconnection, revisited	411
10.5	Ambipolar diffusion	414
	10.5.1 Overview and motivation	414
	10.5.2 A two-fluid, non-isothermal model for AD	416
Proh	blem Set 10	428

#### **Appendices**

Α	Esse	ntials of Vector Calculus	443
	A.1	Vector identities	443
		A.1.1 Identities involving dyadics	444
		A.1.2 Vector derivatives of $\vec{r}$	445
	A.2	Theorems of vector calculus	445
	A.3	Orthogonal coordinate systems	446
	A.4	Euler's and the momentum equations	448
	A.5	The Lorentz force	451
в	Esse	ntials of Electrodynamics	453
	B.1	Maxwell's equations	453
	B.2	Electric energy density	454
	B.3	Magnetic energy density	455
	B.4	Resistive energy density	457
	B.5	The Poynting vector	458

D	The Secant MethodD.1Univariate root finderD.2Multivariate root finder	<b>462</b> 462 465
E	Roots of a Cubic	467
F	Sixth-order Runge–Kutta	468
G	Coriolis' Theorem	476
н	The Diffusion Equation	484
Va	ariable Glossary	486
Re	eferences	493
In	dex	500

## Preface

W<sup>HENEVER</sup> a university professor stares down the barrel of a new course preparation, the first question invariably asked is *Is there a text?* For the most part in undergraduate physics, the answer is usually *Yes, and plenty to choose from*. But for a senior undergraduate or beginning graduate course in *magnetohydrodynamics* (MHD), the selection is much narrower.

MHD is a relatively new branch of physics. Developed by Hannes Alfvén during the 1940s, it didn't gain wide acceptance among physicists writ large until the late 1950s culminating in Alfvén's Nobel Prize in 1970.<sup>1</sup> As such, MHD is often touted as a "classical afterthought", the only branch of classical physics introduced *after* Quantum Mechanics with many of its fundamentals – the Riemann problem, magneto-rotational instability, and non-ideal effects – still being worked out in the 1990s and early aughts.

Thus, MHD has not had as long a history as other branches of physics in which textbooks could accumulate, particularly at the undergraduate level. Indeed, MHD has largely been considered a graduate-level subject and, because of this, the vast majority of existing texts specialise in areas such as fusion physics, solar physics, and planetary discs, many written for students already with some familiarity of plasma physics or fluid dynamics.

Another extenuating circumstance is MHD is a divided field whose practitioners – largely plasma physicists and fluid dynamicists – approach the subject in two very different ways. From a plasma physicist's point of view (PoV), an MHD system is the isotropic limit of an ensemble of charged particles – a plasma – governed by velocity moments of the *Vlasov-Boltzmann equation*, a 6-D inhomogeneous partial differential equation (PDE) at the heart of plasma physics. From a fluid dynamicist's PoV, an MHD system is never considered as an ensemble of particles, but rather as a continuous medium governed by simple conservation rules that *any* undergraduate physics student can understand. This leads to a hyperbolic set of equations that can be analysed entirely in terms of *waves*. The two approaches couldn't be more different.

After thirty years of teaching graduate and undergraduate (astro)physics, it is my considered opinion that for MHD to be approachable by undergraduates, it needs to be taught from the fluids PoV. Wave mechanics – so fundamental in classical mechanics, electrodynamics, and quantum mechanics – is already ingrained in the mind of a fourth-year student. On the other hand, velocity moments of a six-

 $<sup>^{1}</sup>$ On pages 127–128, there's an amusing anecdote on what – or who – changed the physics community's collective mind on MHD, and a link to Anthony Peratt's short biography of Alfvén.

dimensional PDE are not. Further, most texts taking the plasma PoV are focused on laboratory plasmas and fusion physics, and the wave nature of MHD is often overlooked. To my taste, MHD is *the* prototype for teaching and reinforcing *wave mechanics*, and this is precisely the approach I take in this text.

While there are plenty of textbooks on MHD from the plasma PoV, precious few exist from the fluids PoV. In the survey of texts I did as part of my proposal for this text, I found more than 100 books written over the past six decades from the plasma PoV focused on plasma physics with a portion – sometimes substantial – devoted to the "MHD limit". Indeed, a dozen or so of these include MHD in their titles.<sup>2</sup> By contrast, I found *two* texts on MHD written entirely from the fluids PoV and directed to senior undergraduates: Kendall & Plumpton's *Magnetohydrodynamics with Hydrodynamics* (1964); and Galtier's *Introduction to Modern Magnetohydrodynamics* (2016).

This text offers a third. My approach focuses on the *fundamentals* of the subject and teaches MHD for its own sake rather than dwelling on directed applications and current areas of research; these, I argue, are better suited for graduate texts of which there are plenty. I do provide numerous examples from the literature, but these are selected to emphasise certain ideas (*e.g.*, planetary discs with non-ideal MHD, stellar winds with steady-state MHD, astrophysical jets with Bernoulli's principle, *etc.*) and none should distract the reader from the current discussion. Once endowed with the fundamentals, I contend, students can carry these forward to further their study at the graduate level, should they choose.

In keeping with the undergraduate theme, the first part -1-D MHD in Ten Weeks – is designed around a single goal: solving the 1-D MHD Riemann problem. I also assert that to understand MHD, one first has to understand ordinary hydrodynamics (HD) which is, after all, just the zero-field limit of MHD. To these ends, Chap. 1 introduces the student to the fundamentals of HD that includes a novel and simple derivation of the three ideal HD equations. Chapter 2 focuses on 1-D applications of HD including sound waves, shocks, bores, and Bernoulli's principle while Chap. 3 develops a semi-analytic solution to the hydrodynamical Riemann problem. In so doing, students learn how the equations of HD lead to a wave equation, and are shown three ways to extract information about hydrodynamical waves: direct solution of the wave equation; normal mode analysis using linear algebra; and via Riemann invariants and their characteristic paths. In my experience, introducing students to these methods – particularly the latter two – for the relatively simple case of HD is critical for them to understand how they apply to the much more complicated MHD case.

The magnetic induction,  $\vec{B}$ , doesn't appear until Chap. 4 where the ideal induction equation and the Lorentz force are introduced, along with Alfvén's theorem, magnetic helicity, and flux linking. Chapter 5 examines the MHD equations in 1-D to uncover all three types of waves (slow, Alfvén, fast) and all discontinuities (tan-

 $<sup>^{2}</sup>$ The most ambitious and a very recent example of this is Goedbloed, Keppens, & Poedts' Magnetohydrodynamics of Laboratory and Astrophysical Plasmas (2019). This is a comprehensive tour de force which could support at least three graduate-level courses.

gential, rotational, shocks) including all three shock subtypes (slow, intermediate, fast). Chapter 6 introduces slow and fast rarefaction fans, and then brings it all together to show students how an exact MHD Riemann solver can be assembled. As this is a semi-analytical solution, students learn or have reinforced semi-analytical techniques including Runge–Kutta methods, multivariate secant root finders, methods for maintaining machine accuracy, and the list goes on.

Part I is designed to be completed in twenty-five hours of instruction (ten weeks at most Canadian universities). The four chapters in Part II, Additional Topics in (M)HD, are independent from each other, depend only on material from Part I, and give the instructor options to complete the semester. These include (M)HD instabilities in Chap. 7, viscid HD (Navier–Stokes equation) in Chap. 8, steady-state MHD in Chap. 9, and non-ideal MHD in Chap. 10. In the interest of expediency, sections designated as "optional" can be omitted without loss of continuity.

Parts of the text may come across as "mathematically dense"; this is deliberate. As an undergraduate, I always found it frustrating and distracting when I was unable to fill in the large gaps left between lines of logic in the texts my professors chose, and I was not going to produce a text that did the same. That said, the densest parts of the mathematics can largely be skimmed on first read and certainly don't need to be covered in detail in class, as the main results from which the physics is extracted are always boxed within each development. For the student like I was who needs to know how the derivations are done, the gaps between the mathematical steps are small enough that a careful second read should suffice.

More than 125 problems – many exploiting "teaching moments" – and several computer projects are distributed amongst the ten chapters' problem sections, each generally digestible by a senior undergraduate or first-year graduate student. Problems without an asterisk can and should be done in a page or less (and often a few lines), one asterisk indicates a two-page solution, two asterisks indicate a three to four page solution, while three asterisks indicate a more involved problem, generally requiring more than five pages and/or a substantive computer program to solve. A complete solution set including the computer projects is available to the instructor upon request.

The eight appendices are designed to remind students of particularly critical material prerequisite to this text. Students who do not recognise, recall, or know how to use any of this material are encouraged to review the relevant material from previous courses. Following the appendices is a glossary of symbols used throughout the text, a list of references, and finally an extensive index.

While this text assumes no previous knowledge of (M)HD, students should have had second- and third-year courses in mechanics, electrodynamics, and thermodynamics. On the math side, students should be fluent in vector calculus (at the level of App. A), adept at solving differential equations including PDEs such as the wave equation, and thoroughly familiar with linear algebra and, in particular, *eigen*algebra. In addition, some experience in scientific computing (algorithm and code development) would be beneficial.

Finally, an acknowledgement of the biases of the author is in order. While

this text includes numerous astrophysical applications, astronomy is by no means a prerequisite, nor is this text designed just for budding astrophysicists. I would like to think *any* physicist interested in learning about fundamental MHD will find this book useful. As for units, I follow the bulk of the physics community (but not astronomy!) and use mks exclusively. Lastly, all program listings in this text are in FORTRAN77, the *only* computer language – this old programmer would assert – a computational scientist really needs to know!

Like many of my contemporaries, I learned MHD "at the knee of my advisors", by reading select chapters in certain texts, by going through journal articles, and talking to experts. As a student and post-doctoral fellow, it always struck me as a bit unfair that all other branches of physics seemed to be taught in more systematic and accessible ways – dedicated courses, self-contained textbooks, problem sets at appropriate levels – but somehow not MHD. Granted, MHD doesn't enjoy the same "critical mass" of students as other areas of physics, and perhaps this inaccessibility is part of the reason why.

This text – almost two decades in the making – is the textbook I wish I had access to forty years ago when I started out in this game. And now as I enter my retirement, it is my profound hope that within these pages, new students of MHD will find a self-contained introduction to the subject that will help launch them into a fascinating, life-long adventure as I have enjoyed!

Each chapter in Part II benefitted from written projects or theses submitted by Saint Mary's graduate and undergraduate students taking my (M)HD course and/or working with me as a research student in years past. For these efforts, I thank Joel Tanner, Patrick Rogers, Jonathan Ramsey, Nicholas MacDonald, Michael Power, and Christopher MacMackin.

I thank my editors Nicholas Gibbons, Sarah Armstrong, Stephanie Windows, and Jane Chan at Cambridge University Press for their capable and patient guidance of this first-time author. It definitely made my job a lot less daunting! A big thankyou goes to Patricia Langille at Saint Mary's Patrick Power Library for doing *all* the heavy lifting in getting permissions for the copyrighted material used in this text; she gets the first signed copy! I would also like to acknowledge the academic freedom afforded to me over the past three decades by Saint Mary's University that made long-term projects like this possible. Thank you all.

This text was typeset using Donald Knuth's T<sub>E</sub>X and Leslie Lamport's I<sup>A</sup>T<sub>E</sub>X. Many of the figures were created using Xfig developed by Supoj Sutanthavibul, Ken Yap, Brian V. Smith, and others. Figures from *ZEUS-3D* simulations were created using PSPlot developed by Kevin E. Kohler. Countless members of the scientific community are indebted to these people for placing their software into the public domain.

And then there are the magnificent villages of Ménerbes and Saint-Pierre-

Toirac, France. The most difficult sections of this text for me to write were inspired and completed during my long *séjours* there and, other than my home province of Nova Scotia, I can't think of anywhere else I'd rather spend months at a time than in a small French village. So much about France charms me including, of all things, their speed-limit signs! It's not enough to tell you what the speed limit is, the French also feel the need to tell you you're being *reminded* of what the speed limit is! So in homage to my second country, all footnotes throughout the text serving as a "reminder to the reader" are heralded by the French translation *Rappel*.



And finally to my wife Jodi (MEL) to whom this text is dedicated. Words can't express my love and gratitude for sticking by me these nearly forty years and for helping create such a wonderful and supportive home for our family here in Halifax. You're the best!

Any constructive feedback on what works and doesn't work in this text, as well as any error reports, omissions, redundancies, *etc.* are welcome and can be sent to me directly at AfciMHD@gmail.com. Instructors of courses based on this text may download a solution set to the problems and a fully-developed set of course lecture notes from CUP's website, www.cambridge.org/9781009381475.

David Clarke Halifax, Nova Scotia www.ap.smu.ca/~dclarke

## The Fundamentals of Hydrodynamics

Everything flows and nothing abides; everything gives way and nothing stays fixed.

Heraclitus (c. 535-c. 475 BCE)

#### 1.1 Definition of a fluid

The PHYSICS of hydrodynamics (HD), namely conservation of mass, conservation of energy, and Newton's second law, are all concepts familiar to first-year undergraduate students, though the mathematics to solve the relevant equations is not. Consider an *ensemble of particles* within some volume V, and let these particles interact with each other via elastic collisions. We can let V remain fixed (in which case we allow the particles to collide elastically with the walls of the container too), or we can let V increase or decrease as the particles move apart or come together; it does not matter. If the mass, total energy, and momentum of the ensemble of particles are M,  $E_{\rm T}$ , and  $\vec{S}$  respectively, then we have:

$$\frac{dM}{dt} = 0, \qquad \text{conservation of mass;} \qquad (1.1)$$

$$\frac{dE_{\rm T}}{dt} = \sum \mathcal{P}_{\rm app}, \qquad \qquad \text{conservation of total energy;} \qquad (1.2)$$

$$\frac{d\vec{S}}{dt} = \sum \vec{F}_{\text{ext}}, \qquad \text{Newton's second law.} \qquad (1.3)$$

Here,  $\sum \mathcal{P}_{app}$  is the rate at which work is done (power) by all forces *applied* to the ensemble of particles, and  $\sum \vec{F}_{ext}$  are all forces *external* to and acting on the ensemble of particles. Note that the applied forces – normally just collisions from neighbouring ensembles of particles – are typically a subset of the external forces, which include collisions from neighbouring particles *plus* forces arising from gravity, magnetism, radiation, *etc.* This is because in addition to the thermal and kinetic energies, the *total energy*,  $E_{\rm T}$ , includes gravitational, magnetic, radiative, and possibly other energies as well.

It is how we model the collisional forces from neighbouring ensembles of particles that defines both what constitutes a fluid and how Eq. (1.1)–(1.3) are further developed. Consider a small cube with volume  $\Delta V = (\Delta l)^3$  as shown in Fig. 1.1*a*.



**Figure 1.1.** *a*) A single particle bounces elastically from the walls of a cube of edge length,  $\Delta l$ , imparting impulses  $J_x$ ,  $J_y$ , *etc. b*) An *x*-*y* cut through the cube in panel *a* showing one particle whose motion is entirely in the *x*-direction.

Let the walls of the cube be perfectly reflecting and let there be just one particle inside the cube moving at some speed v in an arbitrary direction.

When the particle collides with the wall, both the particle and cube suffer a change in momentum in a direction normal to the surface of the cube. Moments later, the particle collides with a different wall, and the particle and cube suffer changes in momentum in a direction normal to that wall. A change in momentum is an impulse, J, which when multiplied by the time over which the collision occurs,  $\Delta t$ , constitutes the average force. Thus formally, the "pressure", p, the collision exerts on the wall of the box is this average force divided by the area of the wall:

$$p \sim \frac{J\Delta t}{(\Delta l)^2}.$$

In this scenario, the "pressure" is highly variable in time, and by no means could the "pressure" be construed as isotropic. At a given time, the "pressure" one wall feels will have nothing to do with the "pressures" felt by the other walls.

However, by arbitrarily increasing the number of particles,  $\mathcal{N}$ , inside our small volume,  $\Delta V$ , the number of collisions with a given wall, n, occurring in a time  $\Delta t$  will be the same at each wall to within some arbitrarily small variance,  $\Delta n$ . Put another way, averaged over  $\Delta t$ , particle collisions exert the same "pressure" on each wall to within a variance made as small as we please by making  $\mathcal{N}$  as large as we please. Thus, we have rendered the particle "pressure" inside the cube *isotropic* because each wall now feels the same force.

There is a contrived exception to this picture. If all the particles were to be placed initially on the mid-plane of the cube and all were launched with the same speed towards one wall of the cube, then it is only with this and the opposite wall that particles would ever collide, and they would do so in a highly ordered, periodic fashion. The remaining four walls would, in principle, never feel any collisions, and thus the "pressure" in the cube would not be isotropic even with  $\mathcal{N}$  chosen arbitrarily large. Such a well ordered and well directed ensemble of particles is said to be *streaming* and, as  $\mathcal{N}$  is made larger, it becomes increasingly difficult in practice to maintain streaming motion. Small perturbations will eventually cause one particle to collide with another which in turn collide with others, and the ensuing chain reaction quickly reduces the streaming motion to chaos. Isotropic "pressure" (the same "pressure" measured on each of the six walls) is once again the result.

We can now state the key criterion for an ensemble of particles to be treated as a fluid. If there is a sufficient number of particles inside our box (volume element) of dimension  $\Delta l$  so that the motion of particles within the volume element can always be considered isotropic, then the effect of the collisions of particles against the walls of the volume element (which may be rigid walls, or "soft" walls of neighbouring ensembles of particles) is to exert an isotropic "pressure" against all walls. Since isotropy is maintained by particle–particle collisions within the volume element, we may "mathematise" this criterion as,

$$\delta l \ll \Delta l < \mathcal{L},\tag{1.4}$$

where  $\delta l$  is the mean free path (collision length) of the particles,  $\Delta l$  is the length of one side of our cubic volume element containing an arbitrarily large number of particles, and  $\mathcal{L}$  is the smallest length scale of interest in our physical problem. If Ineq. (1.4) holds, we say the ensemble of particles behaves as a *fluid* or a *continuum*. This assumption is an important one; it allows us to treat the applied forces resulting from collisions – which otherwise could be *extremely* difficult to deal with – in a very simple way, namely as an isotropic "pressure".

#### **1.2 A quick review of kinetic theory**

To now, I have been enclosing the word *pressure* in quotation marks. This is because I haven't yet made the logical connection between particle collisions (and more specifically, the momentum transferred by particle collisions) and what we commonly think of as *pressure*, such as the *barometric* pressure of the air. So, before we examine how Eq. (1.1)-(1.3) become the equations of hydrodynamics (HD) under the assumption that the ensemble of particles behaves as a fluid (when Ineq. 1.4 is valid), let us review how the "pressure" and the "temperature" of a fluid relate to properties of the ensemble of particles. These ideas form the basis of *kinetic theory*, often exposed to students for the first time in a first-year physics course.<sup>1</sup>

Consider a cube whose edges of length  $\Delta l$  are aligned with the x-, y-, and z-axes of a Cartesian coordinate system, as depicted in Fig. 1.1. Returning to our example in the previous section, suppose a single point particle of mass m moves inside the cube with velocity  $v_x \hat{x}$  and collides with the wall whose normal is  $+\hat{x}$ . If collisions are all elastic, then the particle reflects from the wall with a velocity  $-v_x \hat{x}$  and thus suffers a change in momentum of  $\Delta S_x = -2mv_x$ . Conservation of momentum then demands that an impulse of  $+2mv_x$  be imparted against the wall. At a time  $\Delta t = 2\Delta l/v_x$  later, the same particle again collides with the wall, imparting another impulse of  $+2mv_x$  against it. Thus, the rate at which momentum is delivered to the

<sup>&</sup>lt;sup>1</sup>For example, Halliday, Resnick, & Walker (2003).

wall by a single particle is given by,

$$\frac{\Delta S_x}{\Delta t} = \frac{2mv_x}{2\Delta l/v_x} = \frac{mv_x^2}{\Delta l} = \langle F \rangle,$$

where  $\langle F \rangle$  is the average force felt by the wall. Thus, the average pressure exerted by this one particle, defined as force per unit area, is given by,

$$\langle p \rangle = \frac{\langle F \rangle}{(\Delta l)^2} = \frac{m v_x^2}{V},$$

where  $V = (\Delta l)^3$  is the volume of the cube. For  $\mathcal{N}$  particles, we simply add over all particles:

$$p \equiv \sum_{i=1}^{\mathcal{N}} \langle p_i \rangle = \sum_{i=1}^{\mathcal{N}} \frac{m v_{x,i}^2}{V} = \frac{m}{V} \sum_{i=1}^{\mathcal{N}} v_{x,i}^2 = \frac{m \mathcal{N}}{V} \langle v_x^2 \rangle, \qquad (1.5)$$

where each point particle is assumed to have the same mass, m, and where  $\langle v_x^2 \rangle = \sum v_{x,i}^2 / \mathcal{N}$  is the arithmetic mean of the squares of the particle velocities.

For any given particle,  $v^2 = v_x^2 + v_y^2 + v_z^2$  and, for large  $\mathcal{N}$ , one would expect  $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$  since one Cartesian direction shouldn't be favoured over another. Thus,

$$\langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle = 3 \langle v_x^2 \rangle, \tag{1.6}$$

and Eq. (1.5) becomes,

$$p = \frac{Nmv_{\rm rms}^2}{3V}, \qquad (1.7)$$
$$v_{\rm rms} \equiv \sqrt{\langle v^2 \rangle},$$

where,

is the *root-mean-square* (rms) speed of the particles in the volume V. Comparing Eq. (1.7) with the *ideal gas law*:

$$p = \frac{\mathcal{N}k_{\rm B}T}{V},\tag{1.8}$$

(where  $k_{\rm B} = 1.3807 \times 10^{-23} \ {\rm J\,K^{-1}}$  is the Boltzmann constant) yields:

$$T = \frac{mv_{\rm rms}^2}{3k_{\rm B}} \quad \Rightarrow \quad \frac{3}{2}k_{\rm B}T = \frac{1}{2}mv_{\rm rms}^2 = \langle K \rangle, \tag{1.9}$$

where  $\langle K \rangle$  is the average kinetic energy per point particle. Thus, while the *pressure*, p, is a measure of the rate at which momentum is transferred from the particles of the fluid (gas) to, for example, the diaphragm of the measuring device (barometer), the *temperature* (or more precisely  $3k_{\rm B}T/2$ ) is a measure of the average kinetic energy of the particles.

The randomly directed kinetic energy of a system of  $\mathcal{N}$  particles is called its *internal energy*, E, and, for the point particles under discussion, is given by,

$$E = \mathcal{N}\langle K \rangle = \frac{3}{2}\mathcal{N}k_{\mathrm{B}}T.$$

The factor 3/2 is significant and warrants comment. A point particle, as may be found exclusively in a monatomic gas, has three *degrees of freedom* of motion, namely *translation* in each of the three Cartesian directions (Fig. 1.2, left).



Figure 1.2. A point particle (left) has three degrees of freedom for movement, while a "dumb-bell" (right) has five.

From Eq. (1.6), we have  $\langle v_i^2 \rangle = \langle v^2 \rangle/3$  for i = x, y, z, and thus to each (translational) degree of freedom we can associate an internal energy  $E_i = \mathcal{N} k_{\rm B} T/2$ , where  $E = E_x + E_y + E_z = 3 E_i$ .

Now, a diatomic molecule (essentially two point masses connected by a massless rod) has the same three translational degrees of freedom as a monatomic particle *plus* two *rotational* degrees of freedom, namely rotation about each of the two principle axes orthogonal to its own axis (the *x*-axis in Fig. 1.2, right), for a total of five degrees of freedom.<sup>2</sup> Note that spinning about the *x*-axis itself does not constitute a degree of freedom as the moment of inertia about this axis is essentially zero. Because of the *principle of equipartition*,<sup>3</sup> each degree of freedom stores the same amount of kinetic energy, and the internal energy of a diatomic gas must be,

$$E = \frac{5}{2} \mathcal{N} k_{\rm B} T.$$
  
$$E = \frac{1}{\gamma - 1} \mathcal{N} k_{\rm B} T, \qquad (1.10)$$

where  $\gamma = 5/3$  for a monatomic gas,  $\gamma = 7/5$  for a diatomic gas, and  $4/3 \leq \gamma < 7/5$  for molecules more complex than diatomic.<sup>4</sup> One can show that  $\gamma = C_P/C_V$ , the ratio of specific heats of the gas, and that for an adiabatic gas (where heat is neither lost nor gained from the system),  $p \propto \rho^{\gamma}$ , where  $\rho$  is the mass density of the gas.

Thus, in general, we write,

Dividing Eq. (1.10) by the volume of the sample and using Eq. (1.8) gives an expression for the *internal energy density*, e:

$$e = \frac{E}{V} = \frac{1}{\gamma - 1} \frac{\mathcal{N}k_{\mathrm{B}}T}{V} = \frac{p}{\gamma - 1}.$$

Thus, an alternate form of the ideal gas law, and the form most frequently used in

 $<sup>^{2}</sup>$ In principle, there are also two vibrational degrees of freedom which, at "ordinary temperatures", statistical mechanics tells us are insignificant.

<sup>&</sup>lt;sup>3</sup>Left to their own devices, systems will distribute the available energy equally among all possible ways energy can be stored. Thus, for a large number of diatomic molecules randomly colliding with each other and the walls of their container, one would not expect  $m\langle v_x^2 \rangle$  to differ significantly from  $m\langle v_y^2 \rangle$  or  $m\langle v_z^2 \rangle$  any more than it should differ from  $I_y \langle \omega_y^2 \rangle$  or  $I_z \langle \omega_z^2 \rangle$ , where  $I_y$  and  $I_z$  are the moments of inertia about the y- and z-axes respectively.

<sup>&</sup>lt;sup>4</sup>Polyatomic molecules are significantly more complex than diatomic molecules, and the full power of statistical mechanics along with a tensor treatment of its moment of inertia are required to explain the value of  $\gamma$  for any individual molecule.

hydrodynamics, is,

$$p = (\gamma - 1) e, \qquad (1.11)$$

which states that the rate at which momentum is transferred via collisions is proportional to the average kinetic energy density (*i.e.*, per unit volume) of the random particle motion.

Possibly the second most frequently used form of the ideal gas law in hydrodynamics is,

$$p = \frac{\rho k_{\rm B} T}{m},\tag{1.12}$$

which follows directly from Eq. (1.8) noting that  $\rho = \mathcal{N}m/V$ . Finally, from Eq. (1.9) [and replacing the '3' with  $2/(\gamma - 1)$ ], we find:

$$v_{\rm rms} = \sqrt{\frac{2k_{\rm B}T}{(\gamma - 1)\,m}} = \sqrt{\frac{2p}{(\gamma - 1)\,\rho}}.$$
 (1.13)

Thus, the rms speed goes as the square root of the temperature. We shall encounter another characteristic speed of the gas proportional to the square root of the temperature in §2.1.1, namely the sound speed,  $c_s$ . Indeed,  $c_s$  and  $v_{\rm rms}$  arise from essentially the same physics, as will be explained when the sound speed is properly introduced.

#### 1.3 The equations of ideal hydrodynamics

In hydrodynamics, the adjective *ideal* means that internal dissipative forces such as viscosity are ignored. A fluid without (with) viscosity is said to be *inviscid* (viscid). In this chapter, our discussion is exclusively restricted to inviscid flow. Viscid flow is more the realm of terrestrial HD (though there are important applications for astrophysical fluids as well), and is covered in some depth in Chap. 8.

We begin our discussion by defining the adjectives *extensive* and *intensive*. Variables such as mass, volume, and energy which are *proportional* to the amount of substance being measured are *extensive* quantities, while mass density (often just referred to as density), energy density, and temperature are *independent* of the amount of substance being studied and are examples of *intensive* quantities.

To give a precise relationship between extensive and intensive quantities, consider a small sample of substance with volume  $\Delta V$ . For every extensive quantity, Q(V,t), of that sample, we can define a corresponding intensive quantity,  $q(\vec{r},t)$ , such that,

$$q(\vec{r},t) = \lim_{\Delta V \to 0} \frac{\Delta Q(V,t)}{\Delta V} = \frac{\partial Q(V,t)}{\partial V}.$$
(1.14)

This is a *microscopic* description of the system; q may well change from point to point. A *macroscopic* description of the system can be obtained by integrating Eq. (1.14) over a finite volume, V, to recover Q:

$$Q(V,t) = \int_{V} q(\vec{r},t) \, dV.$$
 (1.15)

Note that Eq. (1.15) requires that q be an *integrable* function of the coordinates over the volume V, and thus q can be discontinuous and have poles of order less than unity. On the other hand, Eq. (1.14) requires that Q be a *differentiable* function of V, and thus it must be both continuous and free from any poles of any order. Evidently, differentiability is a more restrictive requirement than integrability, and this observation will have important consequences as we develop the theory further.

We're now ready to introduce and prove a theorem that provides a particularly simple way to derive the equations of hydrodynamics from the conservation laws of Eq. (1.1)-(1.3).

**Theorem 1.1.** Theorem of hydrodynamics.<sup>5</sup> If the time dependence of an extensive quantity, Q, is given by:

$$\frac{dQ}{dt} = \Sigma, \tag{1.16}$$

where  $\Sigma$  represents the possibly time-dependent "source terms" (reasons for Q not being "conserved"), then the evolution equation for the corresponding intensive quantity,  $q(\vec{r}, t)$ , is given by,

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\vec{v}) = \sigma, \qquad (1.17)$$

where  $\vec{v} = d\vec{r}/dt$ ,  $Q = \int_V q \, dV$ ,  $\Sigma = \int_V \sigma dV$ , and where the product  $q\vec{v}$  must be a differentiable function of the coordinates.

Proof:

$$\frac{dQ}{dt} = \Sigma \quad \Rightarrow \quad \frac{d}{dt} \int_{V} q \, dV = \int_{V} \sigma dV,$$

where, in general, the volume element V = V(t) also varies in time. Thus, using the standard definition of the derivative,

$$\begin{split} \frac{d}{dt} \int_{V(t)} q \, dV &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_{V(t+\Delta t)} q(\vec{r}, t+\Delta t) dV - \int_{V(t)} q(\vec{r}, t) dV \right] \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_{V(t+\Delta t)-V(t)} q(\vec{r}, t+\Delta t) dV + \int_{V(t)} q(\vec{r}, t+\Delta t) dV - \int_{V(t)} q(\vec{r}, t) dV \right] \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\Delta V} q(\vec{r}, t+\Delta t) dV + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{V(t)} \left[ q(\vec{r}, t+\Delta t) - q(\vec{r}, t) \right] dV \end{split}$$

<sup>&</sup>lt;sup>5</sup>This theorem is a variant of *Reynolds' transport theorem*, a volume-integral application of the Leibniz formula for the derivative of an integral.

where, as shown in the inset, performing the volume integral over the difference in volumes,  $\Delta V = V(t + \Delta t) - V(t)$ , is the same as integrating over the closed surface,  $\partial V$ , using a volume differential given by  $dV = (\vec{v}\Delta t) \cdot (\hat{n}dA)$ . Thus,



$$\begin{split} \frac{d}{dt} \int_{V(t)} q \, dV &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \oint_{\partial V} q(\vec{r}, t + \Delta t) (\vec{v} \Delta t) \cdot (\hat{n} dA) \\ &+ \int_{V(t)} \lim_{\Delta t \to 0} \frac{q(\vec{r}, t + \Delta t) - q(\vec{r}, t)}{\Delta t} dV \\ &= \oint_{\partial V} q(\vec{r}, t) \vec{v} \cdot \hat{n} dA + \int_{V(t)} \frac{\partial q(\vec{r}, t)}{\partial t} dV \\ &= \int_{V(t)} \nabla \cdot \left( q(\vec{r}, t) \vec{v} \right) dV + \int_{V(t)} \frac{\partial q(\vec{r}, t)}{\partial t} dV \quad \text{(Gauss; Eq. A.30)} \\ &= \int_{V(t)} \left( \frac{\partial q(\vec{r}, t)}{\partial t} + \nabla \cdot \left( q(\vec{r}, t) \vec{v} \right) \right) dV = \int_{V(t)} \sigma(\vec{r}, t) dV \\ &\Rightarrow \int_{V} \left( \frac{\partial q}{\partial t} + \nabla \cdot (q \vec{v}) - \sigma \right) dV = 0. \end{split}$$

As this is true for any V, the integrand must be zero, proving the theorem.  $\Box$ 

Note that q is not the conserved quantity, Q is (at least to within a known source term,  $\Sigma$ ). However, since Q is the volume-integral of q, we'll refer to q as a *volume-conserved quantity*.

The quantity  $q\vec{v} \equiv \vec{f}_Q$  is the *advective flux density* of Q whose units are that of Q times  $m^{-2} s^{-1}$ ; this will require a little unpacking. The *flux*,<sup>6</sup>  $\mathcal{F}_Q$ , of a vector field,  $\vec{f}_Q$ , is a measure of how much  $\vec{f}_Q$  "passes through" a given surface area with arbitrary normal,  $\hat{n}$ . Mathematically,

$$\mathcal{F}_Q = \oint_S \vec{f}_Q \cdot \hat{n} \, dA \quad \text{or} \quad \mathcal{F}_Q = \int_\Sigma \vec{f}_Q \cdot \hat{n} \, dA, \quad (1.18)$$

depending on whether the surface is closed (S) or open  $(\Sigma)$  respectively. Thus, the units of  $\vec{f}_Q$  are those of  $\mathcal{F}_Q$  per unit area, and  $\vec{f}_Q$  can also be interpreted as a *flux* density of  $\mathcal{F}_Q$ . And so,  $\mathcal{F}_Q$  is the flux of  $\vec{f}_Q$  while  $\vec{f}_Q$  is the flux density of  $\mathcal{F}_Q$ .

An advective flux density is more specific to fluid dynamics and refers to some quantity, Q, being advected (*i.e.*, transported) by the flow across a surface at a certain rate. Thus, while  $\vec{f}_Q$  is the flux density of  $\mathcal{F}_Q$  with units of  $\mathcal{F}_Q$  per unit area,  $\vec{f}_Q = q\vec{v}$  is also the advective flux density of Q – the volume integral of q – with units of Q per unit area per unit time. It is the "per unit time" part that triggers the adjective.

Evidently, we have four different types of "fluxes" to keep straight (flux, flux

<sup>&</sup>lt;sup>6</sup>From the Latin *fluxus* or "flow", this term was introduced to physics by Sir Isaac Newton.

density, advective flux, advective flux density) and the literature seems to blur all four; often you'll find any or all of these terms used interchangeably. In this text, while I maintain the distinction between *flux* and *flux density*, I've chosen to drop the adjective *advective* to simplify the language a bit, relying instead on context. If a particular flux/flux density has a "per unit time" aspect to it, it is an *advective* flux/flux density; otherwise just flux/flux density.

Last point before getting to the equations of HD: Eq. (1.16) is an *integral* equation (Q and  $\Sigma$  both being volume integrals of intensive quantities, q and  $\sigma$ ), and thus represents a global statement (valid over a finite sample of the fluid) on the conservation of the extensive quantity, Q. On the other hand, Eq. (1.17) is a differential equation (often referred to as the differential form of Eq. 1.16) and thus represents a local statement (valid at a point) on the conservation of Q, involving the corresponding intensive quantity, q. Global and local forms of an equation are not identical. Because differential equations require the functions to be differentiable, solutions of the differential form of the equations can be more restrictive than those of the integral form where functions need only be integrable. More on this in §1.5.

*Example* 1.1. Let Q = M, the mass of the sample of fluid. Find the evolution equation for the corresponding intensive quantity,  $q = \rho$  (mass density).

Solution: From Eq. (1.1),  $\Sigma = 0 \Rightarrow \sigma = 0$ , and Theorem 1.1 requires that:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0.$$
(1.19)

This is the *continuity equation*; the first equation of HD.  $\Box$ 

*Example* 1.2. Let  $Q = E_{\rm T}$ , the total energy of the fluid sample<sup>7</sup> of mass M:

$$E_{\rm T} = E + \frac{1}{2}Mv^2 + M\phi,$$

where E is the internal (thermal) energy and  $\phi$  is the gravitational potential. Find the evolution equation for the corresponding intensive quantity, the *total energy density*, namely,

$$e_{\rm T} = e + \frac{1}{2}\rho v^2 + \rho\phi,$$
 (1.20)

where once again, e is the internal energy density, whose units  $J m^{-3} = N m^{-2}$  are the same as those for pressure, as expected from Eq. (1.11).

Solution: From Eq. (1.2),  $\Sigma = \mathcal{P}_{app} \Rightarrow \sigma = p_{app}$ , the applied power density interpreted as the rate at which work is done on a unit volume of the fluid sample by all applied forces. Thus, Theorem 1.1 implies:

$$\frac{\partial e_{\rm T}}{\partial t} + \nabla \cdot (e_{\rm T} \vec{v}) = p_{\rm app}. \tag{1.21}$$

<sup>&</sup>lt;sup>7</sup>When we introduce magnetism in Chap. 4, we'll add a magnetic term to  $E_{\rm T}$ .



**Figure 1.3.** *a*) A cube of edge length  $\Delta x$  with external pressure forces acting on the *x*-faces indicated. *b*) An *x*-*y* cut through the cube in panel (*a*) showing both the pressure forces on and motion of the *x*-faces.

As discussed in §1.1, applied forces are collisions of external particles with the fluid sample. Thus, the applied power is the rate at which work is done on the fluid sample by the external fluid as the former expands or contracts within the latter.

To find an expression for the applied power,  $\mathcal{P}_{app}$ , consider a small cube of fluid with dimension  $\Delta x$  in the x-direction and cross-sectional area  $\Delta A = \Delta V/\Delta x$  (Fig. 1.3). The pressure force exerted on the left face of the cube is  $F(x) = +p(x)\Delta A$ and, in time  $\Delta t$ , the left face is displaced by  $v_x(x)\Delta t$ . Thus, the work done on the left face by the external fluid is  $\Delta W_{\rm L} = +p(x)v_x(x)\Delta t\Delta A$ . Similarly, the work done on the right face is  $\Delta W_{\rm R} = -p(x + \Delta x)v_x(x + \Delta x)\Delta t\Delta A$  [since  $p(x + \Delta x)$ and  $v(x + \Delta x)$  are oppositely directed; Fig. 1.3b], and the net work done on the fluid cube is:

$$\Delta W = \Delta W_{\rm L} + \Delta W_{\rm R} = p(x) v_x(x) \Delta t \,\Delta A - p(x + \Delta x) v_x(x + \Delta x) \Delta t \,\Delta A$$
$$= p(x) \,\delta V(x) - p(x + \Delta x) \,\delta V(x + \Delta x),$$

where  $\delta V(x) [\delta V(x + \Delta x)]$  is the small volume change on the left [right] face of the cubic sample of volume  $\Delta V$  by virtue of the motion of the left [right] face. Because of its form, this work is frequently referred to as the "pdV term".

Dividing  $\Delta W$  by  $\Delta t$  gives us the applied power,

$$\mathcal{P}_{\text{app}} = \frac{\Delta W}{\Delta t} = -\Delta A \Delta x \frac{p(x + \Delta x) v_x(x + \Delta x) - p(x) v_x(x)}{\Delta x} = -\Delta V \frac{\Delta (pv_x)}{\Delta x},$$

and thus the applied power density is given by:

$$p_{\mathrm{app}} = rac{\mathcal{P}_{\mathrm{app}}}{\Delta V} = -rac{\Delta(pv_x)}{\Delta x}.$$

Taking into account similar terms in the y- and z-directions, and letting  $\Delta \rightarrow \partial$ , we have:

$$p_{\rm app} = -\nabla \cdot (p \, \vec{v}). \tag{1.22}$$

Substituting Eq. (1.22) into Eq. (1.21) yields:

$$\frac{\partial e_{\rm T}}{\partial t} + \nabla \cdot (e_{\rm T} \vec{v}) = -\nabla \cdot (p \vec{v}),$$

$$\Rightarrow \quad \frac{\partial e_{\mathrm{T}}}{\partial t} + \nabla \cdot \left( \left( e_{\mathrm{T}} + p \right) \vec{v} \right) = 0, \qquad (1.23)$$

the total energy equation and the second equation of ideal HD.  $\Box$ 

Example 1.3. Let  $Q = \vec{S}$ , the total momentum of the fluid sample. Find the evolution equation for the corresponding intensive quantity,  $q = \vec{s} = \rho \vec{v}$  (the momentum density).

Solution: From Eq. (1.3),  $\Sigma = \sum \vec{F}_{ext} \Rightarrow \sigma = \sum \vec{f}_{ext}$ , the external force densities. Thus, Theorem 1.1 requires that:

$$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s} \, \vec{v}) = \sum \vec{f}_{\text{ext}}, \qquad (1.24)$$

where the Cartesian representation of the divergence term is:

$$\nabla \cdot (\vec{s} \, \vec{v}) \,=\, \left( \nabla \cdot (s_x \vec{v}), \nabla \cdot (s_y \vec{v}), \nabla \cdot (s_z \vec{v}) \right)$$

(See §A.4 for other orthogonal coordinate systems.)

For now, we will limit the external force densities to terms arising from pressure gradients and gravity. In Chap. 4, we'll add the Lorentz force, in Chap. 8 viscous stress, and in Chap. 10, drag forces exerted between ions and neutral particles. Starting with the pressure gradient, consider once again the small cube of fluid with edge length  $\Delta x$  and face area  $\Delta A$  in Fig. 1.3*a*. If the pressure at the left and right sides of the cube are respectively p(x) and  $p(x + \Delta x)$ , then the net pressure force acting on the cube in the *x*-direction is given by:

$$F(x + \Delta x) + F(x) = -p(x + \Delta x)\Delta A + p(x)\Delta A = -\frac{\Delta p}{\Delta x}\Delta A \Delta x = -\frac{\Delta p}{\Delta x}\Delta V.$$

Thus, the pressure force density in the x-direction is:

$$f_x = \frac{\Delta F_x}{\Delta V} = -\frac{\Delta p}{\Delta x} \to -\frac{\partial p}{\partial x}$$
 as  $\Delta x \to 0$ .

Accounting for all three components,

$$\vec{f_p} = -\nabla p. \tag{1.25}$$

The gravitational force density,  $\vec{f}_{\phi}$ , is even simpler to derive. If the fluid sample has mass  $\Delta M$ , then the gravitational force on  $\Delta M$  is  $-\Delta M \nabla \phi$ , where  $\phi$  is the local gravitational potential arising from all external masses, including other regions of fluid and distant or embedded point masses (*e.g.*, stars). Thus,  $\vec{f}_{\phi}$  is given by:

$$\vec{f}_{\phi} = -\frac{\Delta M \nabla \phi}{\Delta V} \rightarrow -\rho \nabla \phi \quad \text{as } \Delta V \rightarrow 0.$$
 (1.26)

Substituting both Eq. (1.25) and (1.26) into Eq. (1.24) yields the momentum equation, the third and final equation of ideal HD:

$$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s} \, \vec{v}) = -\nabla p - \rho \, \nabla \phi. \qquad \Box \qquad (1.27)$$

Summary of §1.3: Equations (1.19), (1.23), and (1.27) constitute two scalar equations and one vector equation which, when combined with Eq. (1.11), (1.20), and  $\vec{s} = \rho \vec{v}$  (the constitutive equations), provide a closed system of equations for the fluid flow variables, namely the volume-conserved quantities  $\rho$ ,  $\vec{s}$ , and  $e_{\rm T}$ . This suite of equations comprises our first set of equations of ideal hydrodynamics:

Equation Set 1:	
$rac{\partial  ho}{\partial t} +  abla \cdot ( ho  ec v)  =  0;$	continuity
$\frac{\partial e_{\rm T}}{\partial t} + \nabla \cdot \left( (e_{\rm T} + p) \vec{v} \right) = 0;$	total energy equation
$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s}  \vec{v})  =  -\nabla p - \rho  \nabla \phi;$	momentum equation
$e_{\mathrm{T}} = e + rac{1}{2} ho v^2 +  ho \phi;$	constitutive equation 1
$p = (\gamma - 1)e;$	constitutive equation 2
$ec{s}= hoec{v}.$	constitutive equation 3

The gravitational potential,  $\phi$ , is computed by adding up all the potentials of the contributing point masses, and/or by computing the self-gravitational potential of the gas from the density distribution from Poisson's equation:

$$\nabla^2 \phi = 4\pi G\rho. \tag{1.28}$$

As a PDE, Poisson's equation is qualitatively different from the equations of hydrodynamics. It has no time derivative, spatial derivatives are second order, and Poisson's equation is an example of an *elliptical* PDE rather than the *hyperbolic* PDEs of HD (App. C). Analytical methods for solving Poisson's equation can be found in any intermediate or advanced text on electrodynamics (*e.g.*, Paris & Hurd, 1969; Lorrain & Corson, 1970; Jackson, 1975 to suggest a few), while numerical treatments can be found in widely available resources such as *Numerical Recipes* (Press *et al.*, 1992). We shall not address such methods in this text.

#### 1.4 The internal energy density

Equation (1.23) governs the evolution of the total energy density,  $e_{\rm T}$ . We can eliminate the need for the first constitutive equation by finding an evolution equation for the internal energy density, e, alone, and our approach shall be via thermodynamics.

The combined first and second law of thermodynamics is:

$$TdS = dE + p \, dV, \tag{1.29}$$

where the only new variable being introduced is S, the total *entropy* of the fluid

You don't really understand something until you can compute it.

Michael L. Norman computational astrophysicist

#### 6.1 Overview

**H** ANDS DOWN, the trickiest software I have ever written in my forty years of scientific programming is my program to solve the MHD Riemann problem. It's a venture with zero-divides and near-zero divides around every corner, including the usual and relatively simple-to-solve problems in scalar equations where the denominator gets too close to zero, as well as the much more vexing matrix-vector equations where rows of the Jacobian become zero or near-zero, rendering the matrix equation insoluble or nearly insoluble (*i.e.*, dominated by round-off error). All of these challenges present themselves to those who dare tread forward!

On the plus side, nothing has sealed my own understanding of MHD as has the experience of writing an exact MHD Riemann solver. *Anyone* with serious aspirations of understanding the 1-D MHD problem needs to go through this exercise.

And so let's start with an intuition booster. The precept of the Riemann problem is simple enough. As we did in Chap. 3 and as shown in Fig. 6.1, we start with a left and right state where one state is set completely independently of the other. Before t = 0, the two states are separated by an impenetrable diaphragm, **D**, with one state knowing nothing of the other. Then, at t = 0 the diaphragm is removed, and suddenly the two states can interact. One state forces its way into the other and yet somehow, at any given time t, one must still be able to get from the left state to the right via a unique set of allowed MHD transitions. The question is, how do we determine these transitions?

Let's approach this by considering a "building block" example, as depicted in Fig. 6.2. I'm thinking of the wooden  $\operatorname{Brio}^{\mathrm{TM1}}$  train sets my kids and I used to play with when they were little. As shown in panel *a*, suppose there is a vertical gap between A and B that needs to be spanned, and we may do so only with the pieces that come in our set. We'll allow ourselves the latitude of positioning A and B

<sup>&</sup>lt;sup>1</sup>It should not be lost on the reader that this train set analogy pays homage to one of the seminal papers applying the Riemann problem to computational MHD, namely Brio & Wu (1988).

$\rho_{\rm L}$ , $p_{\rm L}$ , $\vec{v}_{\perp_{\rm L}}$ , $\vec{B}_{\perp_{\rm L}}$	$\rho_{\rm R}, p_{\rm R}, \vec{v}_{\perp_{\rm R}}, \vec{B}_{\perp_{\rm R}}$	ŷ
$v_{x_{L}} \longrightarrow$	$\rightarrow V_{x_{\mathrm{R}}}$	. –
left state	right state	

Figure 6.1. Initial set up for the 1-D MHD Riemann problem. At t = 0, the diaphragm, **D**, is removed, and the two left- and right-states interact with an arbitrary jump in (possibly) all flow variables at **D** as initial conditions.

horizontally as needed, but their vertical displacement must remain fixed. Suppose further that our Brio<sup>TM</sup> pieces – to carry on with the analogy – come in a variety of connectors and colours. Standard Brio<sup>TM</sup> pieces are made of maple or birch, and thus light brown with round male–female track connectors at each end. Let's suppose our set comes in four different colours (with just one side of each piece painted), and that all red pieces have a round male and triangular female connector (panel b), all green pieces have a triangular male and rectangular female connector (panel c), and all blue pieces have a rectangular male and pentagonal female connector (panel d). Meanwhile, there is only one black piece, with two male pentagonal connectors (panel e). Further, the red and blue pieces come as flats, ramps, and jumps with the ramps and jumps coming in an assortment of heights, the green pieces come in flats and jumps, and the one black piece is flat. As an added wrinkle, all red ramps go down from the male connector while all red jumps go up, opposite to the blue



Figure 6.2. The "Brio<sup>TM</sup> train track Riemann problem"; see text for description.

pieces where the ramps go up and the jumps go down from the male connector. Green jumps go up or down. Finally, in whatever configuration we come up with to connect A and B, all coloured sides must be face up.

OK, that's what we have and those are the rules. Now let's play!

The first thing to notice is that owing to the shapes of the connectors on each piece, to get from A to B we'll need to arrange the pieces in a specific order. Starting from A, we'll need a red piece, then a green, blue, black, blue, green, and finally a red piece to attach to B. So a first step would be to separate the pieces by colour.

Now comes the harder part where we have to – presumably by trial and error – start fitting pieces to see what combination gets us contiguously from A to B with all connectors joining flatly. It may be that the manufacturer was clever enough to make it so that for any given height difference between A and B, one and only one set of pieces will, in aggregate, span the jump (*e.g.*, panel f in Fig. 6.2). Or perhaps there are numerous solutions, or maybe even none.

The Brio<sup>TM</sup> game just described is almost a perfect analogy to the 1-D MHD Riemann problem considering just one of the components of  $B_{\perp}$ ,  $B_y$  say. If the vertical distance represents the value of  $B_y$  (positive or negative), then the red pieces are the fast waves coming as either rarefaction fans (ramps) or shocks (jumps), the green pieces are the rotational discontinuities, the blue pieces are the slow waves also coming as rarefaction fans or shocks, and the black piece is the contact across which  $B_y$  is constant. The real problem, of course, is much more complicated than this since we have not just one gap to fit, but seven – one for each of the variables  $\rho$ , p,  $v_x$ ,  $v_y$ ,  $v_z$ ,  $B_y$ , and  $B_z$  – where each piece chosen for  $B_y$ , say, dictates which piece must be used for each of the other variables. Thus, finding one set of pieces to span the  $B_y$  gap doesn't necessarily mean the accompanying pieces for  $\rho$ , p, etc., will span their gaps. This is starting to look a bit like a 7-D Rubik's cube<sup>TM</sup>!

And so, with that bit of discouragement, let's get started!

#### 6.2 Non-linear MHD waves

From the discussion in §5.3, we're familiar with the MHD discontinuities (contacts, RDs, fast and slow shocks) mentioned in the Brio<sup>TM</sup> example. All that remains to work out before tackling the MHD Riemann problem directly are the profiles across the fast and slow rarefaction fans (RF). Now, from our discussion in §3.5.3 on the hydrodynamical Riemann problem, we know – at least in principle – how to determine these. Starting from Eq. (3.24), namely,<sup>2</sup>

$$\mathsf{J}_{\mathrm{p}}|q'_{\mathrm{p}}
angle = u_{i}|q'_{\mathrm{p}}
angle$$

where  $|q'_{\rm p}\rangle$  is the derivative of the ket of primitive variables with respect to its argument ( $\xi_i = x - u_i t$ ) and  $J_{\rm p}$  is the primitive Jacobian matrix (both defined in Eq.

<sup>&</sup>lt;sup>2</sup>*Rappel*: This is the *fifth* time we've seen and used this equation, the last time being Eq. (5.55) in §5.2.2.

5.9), we found the seven eigenvalues (characteristic speeds) of  $J_p$  to be  $u_i = v_x \pm a_f$ ,  $v_x \pm a_x$ ,  $v_x \pm a_s$ , and  $v_x$ , where  $a_f$ ,  $a_x$ , and  $a_s$  are the fast, Alfvén, and slow speeds given by Eq. (5.23)–(5.25). Thus, across each wave,  $|q'_p\rangle$  is proportional to one of the right eigenvectors (eigenkets) of  $J_p$ , namely  $|r_i\rangle$ , and, as given in Eq. (3.35),

$$|q_{\rm p}'(\xi_i)\rangle = w_i(\xi_i)|r_i\rangle$$

where  $w_i$  is an arbitrary proportionality or scaling function of the co-moving coordinate,  $\xi_i$ . As we did in §3.5.3, define  $ds_i = w_i(\xi_i)d\xi_i$  as a differential of a generalised coordinate,  $s_i$ , that varies from 0 on the upwind side of the *i*-wave to  $s_{i,d}$  on the downwind side which can be thought of as the "width" or "strength" of the rarefaction fan. Then,

$$\frac{d|q_{\rm p}(s_i)\rangle}{ds_i} = |r_i\rangle, \tag{6.1}$$

gives us a set of seven coupled, first-order ODEs which we integrate through to its width,  $s_{i,d}$ , to find the profiles (in terms of  $s_i$ ) of each primitive variable across the *i*-wave.

And thus we can delay no longer finding the eigenkets of  $J_p!$ 

#### 6.2.1 Fast and slow eigenkets

The eigenkets of interest here are those associated with eigenvalues  $v_x \pm a_f$  and  $v_x \pm a_s$  respectively, as these describe the fast and slow rarefaction fans. The Alfvén and entropy eigenkets (those associated with eigenvalues  $v_x \pm a_x$  and  $v_x$ ) correspond to the Alfvén and entropy waves, both typically discontinuous in some of the flow variables and thus better handled by the conservative equations (*e.g.*, §5.3.2–5.3.4). Further discussion of these is relegated to Problem 6.1.

Starting with the left-moving fast wave with wave speed  $u_1 = v_x - a_f$ , the associated eigenket,  $|r_1\rangle = |r_f^-\rangle$ , is found by solving the matrix equation (again, see Eq. 5.9 for  $J_p$ ),

$$(\mathsf{J}_{\mathrm{p}} - u_{1}\mathsf{I})|r_{1}\rangle = \begin{bmatrix} a_{\mathrm{f}} & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & a_{\mathrm{f}} & \gamma p & 0 & 0 & 0 & 0 \\ 0 & 1/\rho & a_{\mathrm{f}} & 0 & 0 & B_{y}/\mu_{0}\rho & B_{z}/\mu_{0}\rho \\ 0 & 0 & 0 & a_{\mathrm{f}} & 0 & -B_{x}/\mu_{0}\rho & 0 \\ 0 & 0 & B_{y} & -B_{x} & 0 & a_{\mathrm{f}} & 0 \\ 0 & 0 & B_{z} & 0 & -B_{x} & 0 & a_{\mathrm{f}} \end{bmatrix} \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \\ r_{14} \\ r_{15} \\ r_{16} \\ r_{17} \end{bmatrix} = 0,$$

which yields seven linear equations, one of which is redundant. So let's try ignoring the third one (if for no other reason, because it has the most number of terms), and write:

$$a_{\rm f}r_{11} + \rho r_{13} = 0; \tag{6.2}$$

$$a_{\rm f}r_{12} + \gamma pr_{13} = 0; \tag{6.3}$$

$$a_{\rm f}r_{14} - \frac{B_x}{\mu_0\rho}r_{16} = 0; \tag{6.4}$$

$$a_{\rm f}r_{15} - \frac{B_x}{\mu_0\rho}r_{17} = 0; \tag{6.5}$$

$$B_y r_{13} - B_x r_{14} + a_f r_{16} = 0; (6.6)$$

$$B_z r_{13} - B_x r_{15} + a_f r_{17} = 0. ag{6.7}$$

As  $r_{13}$  appears more often than any other component, let's use that as the pivot (scaling factor). Then, Eq. (6.2) and (6.3) give:

$$r_{11} = -\frac{\rho}{a_{\rm f}}r_{13};$$
  $r_{12} = -\frac{\gamma p}{a_{\rm f}}r_{13},$ 

and multiplying Eq. (6.4) by  $B_x/a_f$  and adding Eq. (6.6) gives:

$$-\frac{a_x^2}{a_f}r_{16} + B_y r_{13} + a_f r_{16} = 0 \quad \Rightarrow \quad r_{16} = -\frac{a_f B_y}{a_f^2 - a_x^2} r_{13}.$$
(6.8)

Similarly, Eq. (6.5) and (6.7) yield:

$$r_{17} = -\frac{a_{\rm f}B_z}{a_{\rm f}^2 - a_x^2} r_{13}.$$
(6.9)

Finally, substituting Eq. (6.8) and (6.9) into Eq. (6.4) and (6.5) respectively gives us:

$$r_{14} = -\frac{B_x}{\mu_0 \rho} \frac{B_y}{a_{\rm f}^2 - a_x^2} r_{13}$$
 and  $r_{15} = -\frac{B_x}{\mu_0 \rho} \frac{B_z}{a_{\rm f}^2 - a_x^2} r_{13}$ .

Bringing these results together, the "minus fast eigenket" is,

$$|r_{\rm f}^{-}\rangle = \psi_{\rm f} \begin{bmatrix} -\rho \\ -\gamma p \\ a_{\rm f} \\ -\frac{B_x}{\mu_0 \rho} \frac{a_{\rm f}}{a_{\rm f}^2 - a_x^2} B_y \\ -\frac{B_x}{\mu_0 \rho} \frac{a_{\rm f}}{a_{\rm f}^2 - a_x^2} B_z \\ -\frac{a_{\rm f}^2}{a_{\rm f}^2 - a_x^2} B_y \\ -\frac{a_{\rm f}^2}{a_{\rm f}^2 - a_x^2} B_z \end{bmatrix},$$
(6.10)

where  $\psi_{\rm f} \equiv r_{13}/a_{\rm f}$  is a "scaling factor" which we'll choose for convenience.

Evidently, the "plus fast eigenket" must be identical to Eq. (6.10) with  $-a_f \rightarrow +a_f$ , and the slow eigenkets are just the fast eigenkets with  $f \rightarrow s$ . Thus, we have

for the four fast and slow eigenkets:

$$|r_{\rm f,s}^{\pm}\rangle = \psi_{\rm f,s} \begin{bmatrix} -\rho \\ -\gamma p \\ \mp a_{\rm f,s} \\ \pm \frac{B_x}{\mu_0 \rho} \frac{a_{\rm f,s}}{a_{\rm f,s}^2 - a_x^2} \vec{B}_{\perp} \\ -\frac{a_{\rm f,s}^2}{a_{\rm f,s}^2 - a_x^2} \vec{B}_{\perp} \end{bmatrix} = \psi_{\rm f,s} \begin{bmatrix} -\rho \\ -\gamma p \\ \mp a_{\rm f,s} \\ \pm \operatorname{sgn}(B_x) \frac{a_x a_{\rm f,s} a_{\perp}}{a_{\rm f,s}^2 - a_x^2} \hat{e}_{\perp} \\ -\sqrt{\mu_0 \rho} \frac{a_{\rm f,s}^2 a_{\perp}}{a_{\rm f,s}^2 - a_x^2} \hat{e}_{\perp} \end{bmatrix}, \quad (6.11)$$

where the last four components have been combined into two 2-D vectors  $\propto \vec{B}_{\perp} = (0, B_y, B_z)$ , and where  $a_x = |B_x|/\sqrt{\mu_0\rho}$ ,  $\operatorname{sgn}(B_x) = B_x/|B_x|$ ,  $a_{\perp} = |\vec{B}_{\perp}|/\sqrt{\mu_0\rho}$ , and  $\hat{e}_{\perp} = \vec{B}_{\perp}/|\vec{B}_{\perp}|$ .

While that may have seemed straight-forward enough, the kicker is the denominator in the components proportional to  $\vec{B}_{\perp}$  ( $\hat{e}_{\perp}$ ). Because the 1-D MHD equations are *not strictly* hyperbolic, their eigenvalues can, at times, be degenerate. In particular, we've already seen (*e.g.*, Eq. 5.66) that if  $\vec{B}_{\perp} = 0$  and  $a_x > c_s$ ,  $a_f = a_x$ and the components  $\propto \hat{e}_{\perp}$  for the fast eigenkets blow up. Similarly, for  $\vec{B}_{\perp} = 0$  and  $a_x < c_s$ ,  $a_s = a_x$  and the components  $\propto \hat{e}_{\perp}$  for the slow eigenkets blow up.

Oops.

Our salvation are the scaling factors  $\psi_{\rm f,s}$ , which we choose not so much to normalise  $|r_{\rm f,s}^{\pm}\rangle$  (which we can't anyway since the components have different units), but to render all singularities removable. It turns out our choices are rather limited since, in addition,  $\psi_{\rm f,s}$  must be chosen such that no eigenket is zeroed out. By what has to be described as a stroke of genius and what we'll spend the next few pages justifying, those introduced by Roe & Balsara (1996) and used by Takahashi & Yamada (2014) are:

$$\psi_{\rm f} = \sqrt{\frac{c_{\rm s}^2 - a_{\rm s}^2}{a_{\rm f}^2 - a_{\rm s}^2}} \quad \text{and} \quad \psi_{\rm s} = \sqrt{\frac{a_{\rm f}^2 - c_{\rm s}^2}{a_{\rm f}^2 - a_{\rm s}^2}}.$$
(6.12)

The reader might wonder why the more "obvious" choice of, say,  $\psi_{\rm f,s} = a_{\rm f,s}^2 - a_x^2$  might not be preferred. However, it doesn't take too long to realise that this would result in  $|r_{\rm f,s}^{\pm}\rangle = 0$  for  $B_{\perp} = 0$ , which would mean no wave at all.

The mathematics of MHD rarefaction fans is littered with landmines (*i.e.*, zerodivides or, when it comes time to do the programming, *near* zero-divides) which can confound even the most seasoned algebraist. In my experience, the cleanest approach is to express everything in terms of the MHD-alphas, and then to use the various identities among them to eliminate all differences in the denominators where singularities can occur.

So to start, let's recast the four identities among the speeds  $a_{\rm f}$ ,  $a_{\rm s}$ ,  $a_x$ ,  $a_{\perp}$ , and  $c_{\rm s}$  as listed in Problem 5.4 (Eq. 5.113–5.116), in terms of the MHD-alphas:

0

2

$$a_{\rm f}a_{\rm s} = c_{\rm s}a_x \quad \Rightarrow \quad \frac{a_{\rm f}^2}{c_{\rm s}^2} \frac{a_{\rm s}^2}{c_{\rm s}^2} = \frac{a_x^2}{c_{\rm s}^2} \quad \Rightarrow \quad \boxed{\alpha_{\rm f}\alpha_{\rm s} = \alpha_x;}$$
(6.13)

$$a_{\rm f}^2 + a_{\rm s}^2 = c_{\rm s}^2 + a_x^2 + a_\perp^2 \quad \Rightarrow \quad \overline{\alpha_{\rm f} + \alpha_{\rm s} = 1 + \alpha_x + \alpha_\perp;} \tag{6.14}$$

$$(a_{\rm f}^2 - c_{\rm s}^2)(a_{\rm f}^2 - a_x^2) = a_{\rm f}^2 a_\perp^2 \quad \Rightarrow \quad (\alpha_{\rm f} - 1)(\alpha_{\rm f} - \alpha_x) = \alpha_{\rm f} \alpha_\perp; \tag{6.15}$$

$$(c_{\rm s}^2 - a_{\rm s}^2)(a_x^2 - a_{\rm s}^2) = a_{\rm s}^2 a_{\perp}^2 \quad \Rightarrow \quad (1 - \alpha_{\rm s})(\alpha_x - \alpha_{\rm s}) = \alpha_{\rm s} \alpha_{\perp}.$$
(6.16)

Here,  $\alpha_{\rm f,s}$  are the fast and slow alphas, first introduced in Problem 5.23 (Eq. 5.124). Next, examine the fast eigenkets which, from Eq. (6.11), we can write as:

$$|r_{\rm f}^{\pm}\rangle = \begin{bmatrix} -\psi_{\rm f}\rho \\ -\psi_{\rm f}\gamma p \\ \mp\psi_{\rm f}a_{\rm f} \\ \pm \operatorname{sgn}(B_x)\chi_{\rm f}c_{\rm s}\frac{a_x}{a_{\rm f}}\hat{e}_{\perp} \\ -\chi_{\rm f}c_{\rm s}\sqrt{\mu_0\rho}\hat{e}_{\perp} \end{bmatrix}, \qquad (6.17)$$

where, in terms of the MHD-alphas,  $\psi_{\rm f}$  (Eq. 6.12) and  $\chi_{\rm f}$  are given by:

$$\psi_{\rm f} = \sqrt{\frac{1-\alpha_{\rm s}}{\alpha_{\rm f}-\alpha_{\rm s}}}; \qquad \chi_{\rm f} = \psi_{\rm f} \frac{a_{\rm f}^2}{a_{\rm f}^2 - a_x^2} \frac{a_{\perp}}{c_{\rm s}} = \sqrt{\frac{1-\alpha_{\rm s}}{\alpha_{\rm f}-\alpha_{\rm s}}} \frac{\alpha_{\rm f}}{\alpha_{\rm f}-\alpha_x} \sqrt{\alpha_{\perp}}. \tag{6.18}$$

Then, since  $c_{\rm s}a_x = a_{\rm f}a_{\rm s}$  (Identity 6.13), and  $c_{\rm s}\sqrt{\rho} = \sqrt{\gamma p}$ , we may write Eq. (6.17) in its most compact form:

$$|r_{\rm f}^{\pm}\rangle = \begin{bmatrix} -\psi_{\rm f}\rho \\ -\psi_{\rm f}\gamma p \\ \mp\psi_{\rm f}a_{\rm f} \\ \pm \operatorname{sgn}(B_x)\chi_{\rm f}a_{\rm s}\hat{e}_{\perp} \\ -\chi_{\rm f}\sqrt{\mu_0\gamma p}\,\hat{e}_{\perp} \end{bmatrix}.$$
 (6.19)

Now,  $\psi_{\rm f}$  and  $\chi_{\rm f}$  are much more tightly coupled than Eq. (6.18) appears to suggest. Squaring  $\chi_{\rm f}$  and using identity (6.15), we get:

$$\chi_{\rm f}^2 = \frac{1-\alpha_{\rm s}}{\alpha_{\rm f}-\alpha_{\rm s}} \frac{\alpha_{\rm f}^2}{(\alpha_{\rm f}-\alpha_x)^2} \alpha_{\perp} = \frac{1-\alpha_{\rm s}}{\alpha_{\rm f}-\alpha_{\rm s}} \frac{\rho_{\rm f}^Z (\alpha_{\rm f}-1)^2}{\rho_{\rm f}^Z \alpha_{\perp}^2} \rho_{\perp}$$
$$= \frac{\alpha_{\rm f}-1}{\alpha_{\rm f}-\alpha_{\rm s}} \frac{(1-\alpha_{\rm s})(\alpha_{\rm f}-1)}{\alpha_{\perp}} = \frac{\alpha_{\rm f}-1}{\alpha_{\rm f}-\alpha_{\rm s}} \underbrace{(\alpha_{\rm f}-1-\alpha_{\rm s}\alpha_{\rm f}+\alpha_{\rm s})}_{\alpha_{\perp}} \frac{1}{\rho_{\perp}},$$

using identities (6.13) and (6.14). Thus,

$$\chi_{\rm f}^2 = \frac{\alpha_{\rm f} - 1}{\alpha_{\rm f} - \alpha_{\rm s}} = \psi_{\rm s}^2 \quad (\text{Eq. 6.12})$$
$$= \frac{\alpha_{\rm f} - \alpha_{\rm s} + \alpha_{\rm s} - 1}{\alpha_{\rm f} - \alpha_{\rm s}} = 1 - \frac{1 - \alpha_{\rm s}}{\alpha_{\rm f} - \alpha_{\rm s}} = 1 - \psi_{\rm f}^2.$$

Not only is  $\chi_f = \psi_s$ ,  $\chi_f$  and  $\psi_f$  are related to each other in the same way as sine and cosine! Problem 6.2 completes the symmetry of these factors by showing that,

$$\chi_{\rm s} \equiv \psi_{\rm s} \frac{a_{\rm s}^2}{a_x^2 - a_{\rm s}^2} \frac{a_\perp}{c_{\rm s}} = \psi_{\rm f},$$
(6.20)

and thus  $\psi_s^2 + \chi_s^2 = 1$  as well.

So, let's use these relationships to simplify the notation further by setting  $\mu = \psi_f = \chi_s$ ,  $\nu = \chi_f = \psi_s$  (and thus  $\mu^2 + \nu^2 = 1$ )<sup>3</sup> and bring the fast and slow eigenkets to our final form:

$$|r_{\rm f}^{\pm}\rangle = \begin{bmatrix} -\mu\rho \\ -\mu\gammap \\ \mp\mu a_{\rm f} \\ \pm \operatorname{sgn}(B_x)\nu a_{\rm s}\hat{e}_{\perp} \\ -\nu\sqrt{\mu_0\gamma p}\,\hat{e}_{\perp} \end{bmatrix}; \qquad |r_{\rm s}^{\pm}\rangle = \begin{bmatrix} -\nu\rho \\ -\nu\gammap \\ \mp\nu a_{\rm s} \\ \mp\operatorname{sgn}(B_x)\mu a_{\rm f}\,\hat{e}_{\perp} \\ \mu\sqrt{\mu_0\gamma p}\,\hat{e}_{\perp} \end{bmatrix}.$$
(6.21)

It remains, then, to settle on a final form for the scaling factors  $\mu$  and  $\nu$  and to demonstrate that their apparent singularities are, in fact, removable. As given by Eq. (6.18),  $\mu$  ( $\psi_{\rm f}$ ) has a singularity (but, as we'll see, removable) at  $\alpha_{\rm f} = \alpha_{\rm s}$  which happens only at the so-called *triple umbilic* where  $\alpha_{\perp} = 0$  and  $\alpha_x = 1$  (and thus  $\alpha_{\rm f} = \alpha_{\rm s} = 1$ ; that is, the fast, Alfvén, and slow speeds are *triply* degenerate). If, for convenience, we let,

$$\delta_x = \alpha_x - 1, \tag{6.22}$$

(and thus  $\delta_x \ge 0$  for  $\alpha_x \ge 1$ ), then the singularity in  $\mu$  occurs when both  $\alpha_{\perp}$  and  $\delta_x$  are zero.

By necessity, the MHD Riemann solver we'll design will be semi-analytic. Thus, its reliance on a computer means that even a removable singularity will trigger a zero-divide at the triple umbilic, and we must therefore remove it manually. To do this, it is most convenient to express  $\mu$  in terms of  $\alpha_{\perp}$  and  $\delta_x$ , those quantities which when simultaneously zero cause the singular behaviour.

Starting with the definition of the slow speed (Eq. 5.23),

$$a_{\rm s}^2 = \frac{1}{2} (c_{\rm s}^2 + a^2 - D) = \frac{1}{2} (c_{\rm s}^2 + a_x^2 + a_\perp^2 - D), \qquad (6.23)$$

where the discriminant, D, is given by,

$$D = \sqrt{\left(c_{\rm s}^2 + a_x^2 + a_\perp^2\right)^2 - 4c_{\rm s}^2 a_x^2},$$

we have,

$$\alpha_{\rm s} = \frac{a_{\rm s}^2}{c_{\rm s}^2} = \frac{1}{2} (1 + \alpha_x + \alpha_\perp - d), \qquad (6.24)$$

where,

$$d \equiv \frac{D}{c_{\rm s}^2} = \sqrt{\left(1 + \alpha_x + \alpha_\perp\right)^2 - 4\alpha_x} = \sqrt{\left(\alpha_\perp + \delta_x\right)^2 + 4\alpha_\perp},\tag{6.25}$$

after a little algebra. This, incidentally, is the most robust form for d one can use for a computer application, since there are no subtractions under the radical. Computers, being of finite precision, can often subtract what ought to be equal values and end up with a small residual of "round-off noise" which can be negative as often as

<sup>&</sup>lt;sup>3</sup>Don't confuse  $\mu$  with  $\mu_0!!$ 



Figure 6.3. Profiles of the fast and slow eigenket "scaling factors",  $\mu$  (left; Eq. 6.26) and  $\nu$  (right; Eq. 6.27) as functions of  $\alpha_x = \delta_x + 1$ , shown for various values of  $\alpha_{\perp}$ , including the limiting case of  $\alpha_{\perp} = 0$  (red $\bigcirc$ ) where  $\mu$  and  $\nu$  are step functions with the discontinuity at  $\alpha_x = 1$  ( $\delta_x = 0$ ), the triple umbilic.

not. Under a radical sign, this would trigger a "floating point exception" or worse, the dreaded "NaN" ("not a number") error messages that cause the program to crash, often without any indication of where the first NaN occurred! Further, even if the difference is not purely round-off noise, subtracting two nearly equal numbers can yield results significantly less precise than the stated precision of the machine. Conversely, adding positive quantities suffers no such loss of machine accuracy.

Continuing from Eq. (6.24), we have,

$$\alpha_{\rm s} = \frac{1}{2} (2 + \delta_x + \alpha_\perp - d) = 1 + \frac{1}{2} (\delta_x + \alpha_\perp - d)$$
$$\Rightarrow \quad 1 - \alpha_{\rm s} = \frac{1}{2} (d - \delta_x - \alpha_\perp).$$

Noting that  $\alpha_{\rm f} - \alpha_{\rm s} = d$ , we arrive at our final forms for  $\mu$  and  $\nu$ :

$$\mu^{2} = 1 - \nu^{2} = \frac{1 - \alpha_{\rm s}}{\alpha_{\rm f} - \alpha_{\rm s}} = \frac{1}{2} \left( 1 - \frac{\delta_{x} + \alpha_{\perp}}{d} \right);$$
(6.26)

$$\nu^{2} = 1 - \mu^{2} = \frac{\alpha_{\rm f} - 1}{\alpha_{\rm f} - \alpha_{\rm s}} = \frac{1}{2} \left( 1 + \frac{\delta_{x} + \alpha_{\perp}}{d} \right), \tag{6.27}$$

with d given in terms of  $\delta_x$  and  $\alpha_{\perp}$  by Eq. (6.25).

Figure 6.3 shows  $\mu$  and  $\nu$  as functions of  $\alpha_x = \delta_x + 1$  for various values of  $\alpha_{\perp}$ , including the limiting case (red $\bigcirc$ ) where  $\alpha_{\perp} = 0$  ( $B_{\perp} = 0$ ). The fact that  $\mu^2$  (and thus  $\mu$ ) should be a step function when  $\alpha_{\perp} = 0$  is easy to see from Eq. (6.25) and (6.26). Setting  $\alpha_{\perp} = 0$ ,

$$d = \sqrt{(\delta_x)^2} = |\delta_x|$$

$$\Rightarrow \quad \mu^2 = \frac{1}{2} \left( 1 - \frac{\delta_x}{|\delta_x|} \right) = \begin{cases} 1, & \delta_x < 0, \ (\alpha_x < 1); \\ 0, & \delta_x > 0, \ (\alpha_x > 1), \end{cases}$$
(6.28)

which is the red $\bigcirc$  profile in the left panel of Fig. 6.3. Evidently, the step function is reversed for  $\nu^2 = 1 - \mu^2$  (0 for  $\alpha_x < 1$ , 1 for  $\alpha_x > 1$ ), as the red $\bigcirc$  profile in the right panel shows.

The fact that  $\mu$  and  $\nu$  remain finite everywhere means that the apparent singularity at the triple umbilic (when  $\alpha_{\rm f} = \alpha_{\rm s}$  in Eq. 6.26) is removable; that is,  $\mu^2$ remains finite as  $\alpha_{\rm f} \to \alpha_{\rm s}$  (when both  $\alpha_{\perp}$  and  $\delta_x$  are zero). To find that limiting value, we first set  $\delta_x = 0$  in Eq. (6.25) and (6.26) to get:

$$\lim_{\delta_x \to 0} \mu^2 = \frac{1}{2} \left( 1 - \frac{\alpha_\perp}{\sqrt{\alpha_\perp^2 + 4\alpha_\perp}} \right) = \frac{1}{2} \left( 1 - \sqrt{\frac{\alpha_\perp}{\alpha_\perp + 4}} \right) \to \frac{1}{2}, \tag{6.29}$$

as  $\alpha_{\perp} \to 0$ . Similarly,  $\nu^2 \to \frac{1}{2}$  as  $\delta_x$ ,  $\alpha_{\perp} \to 0$ .

This completes the justification of the scaling factors,  $\psi_{f,s}$ , chosen in Eq. (6.12).

#### 6.2.2 Fast and slow rarefaction fans

Part of the job of the Riemann solver is to integrate Eq. (6.1) using the kets in Eq. (6.21) to find the primitive variable profiles across any rarefaction fan that may be part of the solution. However, without doing the actual integrations, we can determine qualitative properties of the fast and slow fans just by examining the kets as differential changes in the variables.

Assuming for now that  $B_x \neq 0$  ( $\alpha_x > 0$ ), let's start by examining the first three components of each ket. Other than the factors  $\mu$  and  $\nu$ , these are identical to the purely hydrodynamical kets in Eq. (3.34), with  $c_s$  replaced with the appropriate wave speed. So, for the moment, let's set  $B_{\perp}$  (and thus  $\alpha_{\perp}$ ) to zero so that for  $\alpha_x < 1$ ,  $\mu = 1$  and  $\nu = 0$  (Fig. 6.3). This makes the fast kets *identical* to the hydrodynamical kets for which, in §3.5.3, we concluded that density and pressure *decrease* from the upwind to downwind side of the fan, while the flow speed relative to the upwind state *increases*. A similar comparison between the slow and hydrodynamical fans may be made for  $\alpha_x > 1$  where  $\nu = 1$  and  $\mu = 0$ .

Noting from Fig. 6.3 that the primary effect of increasing  $\alpha_{\perp}$  from zero is to round off the discontinuities in  $\mu$  and  $\nu$  without changing their monotonic dependence on  $\alpha_x$ , we conclude that  $\rho$  and p should decrease across any RF while the flow speed relative to the upwind state increases. Thus, at least qualitatively, the hydrodynamical variables  $\rho$ , p, and  $v_x$  behave the same way across a fast and slow fan as they do across a hydrodynamical fan.

A consequence of p dropping across an MHD fan is that  $\alpha_x = a_x^2/c_s^2 = B_x^2/(\mu_0\gamma p)$  rises. This is an important observation that holds for any MHD RF. Note that  $\alpha_x$  rises only because p falls; in a 1-D problem such as this,  $B_x$  is strictly constant. Thus, one can gain a qualitative feel for how  $\mu$  and  $\nu$  vary across a fan – quantities critical to determining the variable profiles from Eq. (6.1) – by scanning across Fig. 6.3 from left to right.

The last component in each ket in Eq. (6.21) governs the profiles of  $B_{\perp}$ , and it is here where the properties of the fast and slow fans diverge. For the fast fan, the last component,  $-\nu \sqrt{\mu_0 \gamma p}$ , is negative – just like the first two components governing the

### Index

 $1\frac{1}{2}$ -D flow, definition, 34, 123

abbreviated Leibniz notation, 123, 295 accretion discs, see planetary discs Alfvén, Hannes, ii, xiii, 102, 127–128 Alfvén number, **156**, 157, 165, 344 Alfvén point, **165**, 166–170, 179, 344, 346, 349, 352, 360, 371 Alfvén speed,  $a, a_x, 127, 133, 145, 147,$ 148, 156, 186, 344, 360, 387, 413 trans-Alfvénic, 155 Alfvén waves, *see* wave families, MHD Alfvén's theorem, 105–106, 157, 342, 393, 396ambipolar diffusion, 414–427 ambipolar electric field,  $E_{AD}$ , 414 ambipolar resistivity,  $\eta_{AD}$ , 433 coefficients ambipolar,  $\beta_{1,2}$ , 382, 386 coupling,  $\gamma_{1,2}$ , 382 exchange-ambipolar,  $Q_{i,n}$ , 420, 425, 427rate,  $\langle \sigma u \rangle_{1,2}$ , 382, 426, 428–432 density source terms, 417–419 exchange,  $\sigma_{1,2}$ , 417–419, 424, 436, 437 energy source terms, 420–423 ambipolar,  $\varpi_{1,2}^{a}$ , 417, 420–421 exchange,  $\varpi_{1,2}^{x}$ , 417, 421–423 momentum source terms, 419–420 ambipolar,  $\vec{f}_{1,2}^{a}$ , 380–385, 417, 419 exchange,  $\vec{f}_{1,2}^{x}$ , 417, 419–420 two-fluid, isothermal model, 415 two-fluid, non-isothermal model, 416 - 427two-fluid resistivity,  $\eta_2$ , 425 Ampère's law, 106, 454, 456 anti-curl, 112, 118 applied power,  $\mathcal{P}_{app}$ , 7, 16 ambipolar power density,  $p_{AD}$ , 439 density,  $p_{\rm app}$ , 15, 16 electric power density,  $p_E$ , 455 electromagnetic power,  $\mathcal{P}_{\rm EM}$ , 458 magnetic power density,  $p_{\rm M}$ , 457 Poynting power density,  $p_S$ , 458 resistive power density,  $p_{\rm R}$ , 392, 457 astrophysical jets, 62–68, 100, 366–371 Mach disc (hot spot), 62

restarting jet, 63–65 wide-angle tailed sources (WATs), 65–68 AZEuS, 366-371 barotropic gas, 20, 23, 24, 358 Bay of Fundy, 42 lunar resonance, 59 bead-on-a-rod problem, 352-353, 478-483 inertial reference frame, 481–483 non-inertial reference frame, 479–481 Bernoulli levitation, 52–54 air gap, 53 maximum supportable mass, 62 Bernoulli's theorem, HD, 49, 47–55 gas, 49and Kelvin–Helmholtz instability, 252 liquid, 49 Bernoulli's theorem, MHD, 352 Bernoulli function,  $\mathcal{B}_{M}$ , 349–372 critical points, 359-360, 362-364, 375 as function of  $\rho$  and s, 358–359 inertial frame derivation, 372 unitless, 362, 364 value for stellar winds, 361 as a driver for outflow, 352effective potential,  $\phi_{\text{eff}}$ , 356–357, 374 role of B, 352–354, 372 Biot and Savart, law of, 400, 405, 412 bores, 42-47foaming, 43, 45, 46 lab frame, 46-47sluice gate, 61 standing (hydraulic jump), 43, 61 tidal, 42undulating, 43, 45, 46 velocity jump, 45 boundary conditions axisymmetry, 334 Cauchy, 330, 334 fluid–solid (no slip), **327**, 327–330, 336 free boundary, 330 bra-ket notation,  $\langle | \rangle$ , 30 eigenbras (|, 84–85, 96, 175, 236 eigenkets  $|\rangle$ , 83–85, 96 Brio & Wu problem, 160, 161, 183 broad-crested weir, 49–52

flowrate, 51

Cardano–Tartaglia formula, 163, 467 properties of cubic roots, 169 characteristic paths, 74, 76, 78, 80, 128-129, 135, 137-139, 177, 202-203, 210characteristic speeds, 71, 73–74, 86, 461 1-D MHD, 128, 186 Alfvén waves, 135 magnetosonic waves, 148 characteristics Alfvén waves,  $A^{\pm}$ , 135, 136, 177 calculating, 85 entropy wave,  $S^0$ , **73–74**, 75, 76, 86 fast/slow waves,  $\mathcal{J}^{\pm}$ ,  $\mathcal{S}^{\pm}$ , 201 sound waves,  $\mathcal{J}^{\pm}$ , **73–74**, 76, 201 where  $\mathcal{J}^{\pm}$ ,  $S^0$  are constant, 86, 97 circulation,  $\Gamma$ , 24, 117 colon product of matrices, A : B, 321, 444 combined laws of thermodynamics, 18, 38 conics of PDEs, 459-461 conservation laws for MHD, 106 energy, 7 magnetic flux, 106 mass, 7Newton's second law, 7 conservative variables, 22 contact discontinuity, HD, 35, 258 isothermal, 57 polytrope, 58 contact discontinuity, MHD, 154, 202 continuity equation, 15 incompressible flow, 43, 258 linearised, 293 steady state, 48, 342 control volume (steady-state HD), 43 cosmic rays, 100, 286-287 pressure equation, 292 linearised, 294 pressure,  $p_{CR}$ , 288, 289, 305 Crab nebula, 257 current density,  $\vec{J}$ , 106, 143, 383, 408, 424, 435, 456 current sheet, 395, 435 de Laval nozzle, 54–55, 62 choke point, 55de Laval's equation, 55 in radio source 1919+479, 66 difference theory, 150–151, 178 diffusion equation, 327, 484–485 diffusion coefficient, 327, 485 Fick's first law, 484 diffusion time scale, 327 magnetic, 393, 395 dimensional analysis, 51–52 discharge rate flow between plates, 329, 339

Hagen–Poiseuille flow, 334 open channel flow, 331 discontinuous flow, 22, 90 downwind; definition, 34 dyadic product, 22, 111, 444 dynamos, 399, 399-406 anti-dynamos, 400-402, 434 Bullard dynamo, 399-400 Earth's dynamo, 403-406, 434 magnetic pole drift, 403polarity flips, 406 Taylor column, 405 kinematic vs. non-linear, 401 necessary and sufficient conditions, 403 Earth's magnetic field, **3**, 403–406, 434 eigenvalues, HD, 85 sound waves, 31 eigenvalues, MHD, 127–128, 172, 174, 186 Alfvén waves, 132 magnetosonic waves, 140–141 eigenvectors, HD rarefaction wave, 86, 95 sound waves, 31 eigenvectors, MHD Alfvén waves, 132, 186, 235 entropy wave, 186, 235magnetosonic waves, 141–145, 186–190 normalisation,  $\psi_{\rm f}$ ,  $\psi_{\rm s}$ , 188, 236 scaling factors,  $\mu$ ,  $\nu$ , 190–192, 195, 220, 236, 238 electric energy density,  $e_E$ , 455 electric field,  $\vec{E}$ , 453 ambipolar diffusion,  $E_{\rm AD}$ , 414 Hall,  $\vec{E}_{\rm H}$ , 408 non-ideal MHD fluid, 385 resistive,  $E_n$ , 392 static vs. induced, 102 supported by a conducting medium, 102electromagnetic force,  $F_{\rm EM}$ , 101, 391, 454 density,  $\vec{f}_{\rm EM}$ , 381 elliptical equations, 18, 459 energy, see internal, total, or magnetic energy enthalpy, h, 49, 351, 356 entropy, S, 18, 38, 73 per particle, S, 19, 23 specific, s, 19 equations of HD, 18, 20, 21, 71 conservative form, 18, 80 1-D steady state, 34 differences between forms, 20–22 Eulerian form, 71 Lagrangian form, 71 primitive form, **21**, 80 in 1-D, 81 1-D general solution, 84, 96

conservative form, 111 1-D steady state, 149 in 1-D, **124**, 172 for Parker instability, 292 in most compact form, 112 primitive form, 122 1-D linearised, 140, 173 in 1-D, **124**, 172 steady-state, 342 equations of MHD (non-ideal) neutrals, ions, electrons, 416 one-fluid, isothermal model, 386 limitations, 390 one-fluid, non-isothermal model, 438 three non-ideal terms, 379 their comparison, 387–391 two-fluid, isothermal model, 427 two-fluid, non-isothermal model, 427 equilibrium, stable vs. unstable, 243 Euler number,  $\mathcal{E}$ , 324 Eulerian reference frame, 71, 280 space-time diagrams, 74, 76, 77 Euler's equation, HD, **20**, 22, 26, 322 1-D, 72 linearised, 27, 29, 247, 260 orthogonal coordinate systems, 448–450 scaled version, 322-323, 338 Euler's equation, MHD, 107, 177 steady state, 342, 398 evolutionary vs. non-evolutionary, 160 extensive vs. intensive var., 12, 103 relationship between, 12 Faraday's law, 102, 385, 392, 426, 454 fast point, 165, 169, 180, 354, 360, 362-364, 375 fast speed,  $a_{\rm f}$ , see magnetosonic speeds flowline, 48 fluid, definition, 2, 7–9 impedance,  $Z_0$ , 32 inviscid, 12, 25 viscid, 12 fluid dynamics, definition, 2fluid mechanics, definition, 2 flux, flux density, 103 definitions, 14 HD flux densities, 22 linearised, 30 MHD flux densities, 125 flux function, f, 342–345, 358, 400 coordinate, s, 350as lines of induction,  $\vec{B}$ , 342 twisting lines of induction, 344-345 flux loop, 114 Flux theorem, 103–105, 117 flux tube, 105, 347-348

equations of MHD (ideal), 109, 111

flux-freezing, 105, 157, 342, 396 flux-linking, 115 on solar surface, 116 force densities,  $\vec{f}_{ext}$ , 17 ambipolar,  $\vec{f}_{1,2}^{a}$ , 380 exchange,  $\vec{f}_{1,2}^{x}$ , 420 gravity,  $\vec{f}_{\phi}$ , 17 Lorentz,  $\vec{f}_{\rm L}$ , 107 pressure gradient,  $\vec{f_p}$ , 17 viscous stresses,  $f_{\rm T}$ , 318 force-free condition, 357, 374 forces applied vs. external, 7 collisional, 7-10 Froude number,  $\mathcal{F}$ , 263, 338 gas dynamics, definition, 2 Gauss' law, 454 Gauss' theorem, 14, 21, 44, 397, 403, 445, 457generalised Ohm's law, see electric field,  $\vec{E}$ ; non-ideal MHD fluid Green's theorem, 445 Hall MHD, 406–414, 435 Hall current, 410 Hall effect (lab), 407–408 Hall effect (plasma), 408 Hall electric field,  $\vec{E}_{\rm H}$ , 408, 426 magnetic reconnection, 411-414 proton–electron resistivity,  $\eta_{p,e}$ , 410 quadrupole magnetic moment, 413 two component model, 408-410 helicity, see magnetic helicity helicity flux,  $\vec{\mathcal{F}}_h$ , 114 hydrodynamics definition, 2ideal, 12 hyperbolic equations, 82, 83, 128, 459 strictly vs. not strictly, 82, 128, 202 ideal gas law, 10-12, 289, 421 induction equation ideal, 102, 109-111, 177 cf. vorticity equation, 118 linearised, 294 steady state, 342 non-ideal, 385 ambipolar diffusion, 386 Hall, 386, 408, 426 resistive, 386, 391–392, 394, 400, 402 two-fluid, 426 instabilities, see KHI, RTI, Kruskal, MRI, Parker intensive var., see extensive vs. intensive var.

intermediate point, 166, 169, 170 internal energy, E, 10–11 density, e, 11 specific,  $\varepsilon$ , 19, 73 internal energy equation adiabatic, 20, 23 isothermal, 23 one-fluid, ambipolar, 439 resistive, 393 viscid form, 321, 337 interstellar medium (ISM), 286–288 Jacobian matrix, HD conservative, 95 primitive, 82 sound waves, 30Jacobian matrix, MHD Alfvén waves, 131 conservative, 125–126, 172 magnetosonic waves, 140 primitive, 124, 127, 172, 186 Riemann problem, see Riemann problem, MHD; Jacobian Jupiter's Great Red Spot, 253–254 Kelvin–Helmholtz instability (KHI), 245-255, 325and Bernoulli's theorem, 252 cat's eves, 252 condition for instability, 249 dispersion relation, 248–249 growth rate, 250 linear vs. non-linear theory, 251–252 normal mode analysis, 246–250 numerical analysis, 251-252, 254-255 slab jet, 305Kelvin's circulation theorem, 24, 117, 258 kinetic energy density, k, 402 flux,  $\vec{\mathcal{K}}$ , 375 Kruskal–Schwarzchild instability, 267 dispersion relation, 267 Lagrangian derivative, 71, 73, 135, 177, 401 Lagrangian reference frame, 71, 280 space-time diagrams, 74, 75 Lagrangian velocity, 71 laminar flow, 325–336 Laplace's equation, 25, 259, 374 pseudo-Laplacian operator,  $\nabla^2$ , 400 Larmor radius,  $r_{\rm L}$ , 100 liquid, definition, 2Lorentz force,  $\vec{F}_{\rm L}$ , 3, 99, 106, 289, 396, 400, 408 density,  $f_{\rm L}$ , 106, 143, 145, 270, 373, 383, 424longitudinal terms, 111

orthogonal coordinate systems, 451–452 transverse terms, 111 LU decomposition, 213–214 Lundquist number, S, 387, 398 Mach number, M, **36**, 55, 180 downwind of shock, 37transonic point, 37, 40, 55 upwind of shock, 157 magnetic diffusion, 393 magnetic diffusivity,  $\mathcal{D}_{\rm M}$ , 393, 403 magnetic energy,  $E_{\rm M}$ , 403 density,  $e_{\rm M}$ , 111, 402, 457 magnetic field,  $\dot{H}$ , 3, 453 in astrophysics, 99–100 potential field, 374 magnetic flux,  $\Phi_B$ , **105–106**, 110 conservation of, 105–106, 115, 157, 361 magnetic helicity,  $H_A$ , 113–114 as a conserved quantity, 114 cross helicity,  $h_{\times}$ , 122 density,  $h_A$ , 113–114 evolution equations, 113–114 value in flux loop(s), 115 magnetic induction,  $\vec{B}$ , 3, 453 magnetic reconnection, 393-399, 411-414Hall regime, 412 quadrupole magnetic moment, 413 reconnection time scale, 413 Sweet–Parker model, 395–399 reconnection time scale, 398 X-point, 395, 398, 399, 412 magnetic topology, 114–116 magnetic torque/moment, 347-348 torque density, 347 magneto-acoustic waves, 145, 149, 201, 202 speed,  $a_{\rm M}$ , 145, 195 magnetohydrodynamics (MHD) definition, 2ideal, definition, 101 magneto-rotational instability (MRI), 268 - 286angular momentum transport, 278–307  $\dot{L}$  transported per revolution, 283 Balbus & Hawley, Shaw Prize, 268, 278 comparison to KHI, 274, 285 condition for instability, 274–275 diffusion coefficient, 307 dispersion relation, 273 dynamical equations, 273 growth rate, 277–278, 306 normal mode analysis, 272–274 numerical analysis, 283-286 physical model, 275–277 magnetosonic numbers,  $M_{\rm f}$ ,  $M_{\rm s}$ , 165, 169, 368

magnetosonic speeds,  $a_{\rm s}$ ,  $a_{\rm f}$ , **127**, 141, 148, 186, 360 identities, 173 inequalities, 128, 173 limits, 174 Maxwell's equations differential form, 453 integral form, 453-454 mean free path, definition, 2method of characteristics (MoC), 76–78 applied to Alfvén waves, 135–176 applied to Riemann problem, 78-80 as a numerical scheme, 77, 94 MHD-alpha, α, 106, **130**, 157, 179, 236, 237, 388, 426 fast, slow,  $\alpha_{f,s}$ , 180, 189, 190, 195 identities, 188 momentum equation, HD, 17, 310 orthogonal coordinate systems, 449–450 momentum equation, MHD Hall, 408 ideal, **107**, 111, 383 ions, two-fluid, 424 linearised, 293, 294 neutrals, ions, electrons, 380 momentum,  $\vec{S}$ , 7 density,  $\vec{s}$ , 17 Navier–Stokes equation, 317–320 compressible, 319 incompressible, 320 scaled version, 323 inertial term,  $\nabla \cdot (\vec{s} \, \vec{v})$ , 318 magnetic, 401 stress force density,  $f_T$ , 317–318 viscid momentum equation, 318 viscid term,  $\nabla \cdot (\mu S)$ , 318 Newtonian fluids, 314–315, 318 non-inertial reference frame, 269–350, 352, 401, **476–483** Coriolis theorem, 478 inertial accelerations, 270-271, 350, 405-406, 478 normal mode analysis Alfvén waves, 131–134, 175 ball on a mound, 305explained, 247 KHI, 246–250 magnetosonic waves, 140–141 MRI, 272-274 Parker instability, 298–301 rarefaction fan, 82 RTI, 257-262 sound waves, 30-33 numerical considerations convergence, 217, 219, 237, 238, 468-469

preserving precision, 180, 191, 208-209, 212-216, 218, 220-221 scaling, 262-264 suppressing pressure perturbations, 284 numerical MHD, 119, 139, 366–371 Ohmic resistance, see resistive MHD outflow mechanisms bead-on-a-rod (BRM), 353-357 critical angle  $(60^{\circ})$ , 355–357, 374 energy flux, 375 magnetic tower (MTM), 354, 373 and MHD Bernoulli theorem, 352 parabolic equations, 326, 327, 459, 485 Parker instability, 286–305 2-D equations of MHD, 292 comparison to RTI, 291 condition for instability, 289, 301-303 dynamical equation, 298 growth rate, 288, 290, 301-302 interstellar clumps, 287–288, 290, 303 normal mode analysis, 298–301 perturbation analysis, 292–294 qualitative description, 286–291 quantitative description, 291-305 particle path, 47 Pascal's law. 318 PdV term, 16 planetary discs, 268, 426 anomalous viscosity, 268 artist's conception, 379 formation, 354–355 number density, 388 T Tauri IM Lup, 378 temperature, 379, 388 plasma physics, definition, 3, 99 plasma-beta,  $\beta$ , **129** Poisson's equation, 18 polytropic gas, 57 power, see applied power Poynting flux,  $\Phi_S$ , 107, 458 power density,  $p_S$ , 107, 109, 458 vector,  $\vec{S}_{\rm P}$ , 107, 375, 458 pressure cosmic ray,  $p_{\rm CR}$ , 288, 289, 305 magnetic,  $p_{\rm M}$ , **111**, 142–145, 270, 288 MHD, p\*, **111**, 271, 279 thermal, p, **17** collisional, 8 isotropic, 8-10 pressure equation, 20, 23 1-D, 72 barotropic, 26 cosmic rays, 292 linearised, 27, 29, 294

pressure head, 44, 259 primitive variables, 22, 34 principle of equipartition, 11, 100, 303 Rankine–Hugoniot, HD, **35**, 80, 90 isothermal, 57 polytrope, 57 Rankine–Hugoniot, MHD, **150**, 153, 157 rarefaction fans, HD, 80 generalised coordinate,  $s_i$ , 86 profiles as function of  $s_i$ , 87 profiles as function of  $u_i$ , 88, 89, 97 strength/width, 87 transition, 87, 89 rarefaction fans, MHD, 185, 192–202, 239 fast fans, 193 fast Euler fans, 193, 196, 200, 201, 204, 207 saturation, 193, 196, 201, 207, 219 switch-off fans, 193, 196, 200, 203, 236 fast/slow differences, 192, 200–202 generalised coordinate,  $s_i$ , 186 profiles as function of  $s_i$ , 186, 218 profiles as function of  $u_i$ , 196–200 similarity to HD fans, 192, 195, 200 slow fans, 193–195 asymptotic limits, 193, 200 slow Euler fans, 193, 195, 200, 201, 204switch-on fans, **194–195**, 198–201 strength/width, 186, 193, 200, 207 ratio of specific heats,  $\gamma$ , 11 Rayleigh–Taylor instability (RTI), 256–267 Atwood number, 261 condition for instability, 261 dispersion relation, 259–261 growth rate, 261 link with KHI, 262, 265, 266 normal mode analysis, 257–262 numerical analysis, 262–267 resistive MHD, 391–406 energy dissipation, 392–393 resistive electric field,  $E_{\eta}$ , 392 resistivity,  $\eta$ , 385 Reynolds number, *R*, **324**, 322–326 inertial vs. viscous dominance, 325 magnetic,  $\mathcal{R}_{M}$ , 387, 403 Riemann, Bernhard, 69 Riemann invariants, see characteristics Riemann problem, HD defined, 69 solution, 90-93, 98 Riemann problem, MHD defined, 183-185 exact solver, 204–221, 239 algorithm, 210-221 constraints, 206, 210-212

fast shock, 215-216 Jacobian, 209-210, 212-214 parameters, 205–208, 210 rarefaction fans, 218–221 slow shock, 216–218 strategy, 208-209 solutions, 221–235 uniqueness, 160, 204 rms speed,  $v_{\rm rms}$ , **10**, 12, 29, 289, 304 rotational discontinuity, 155, 165, 167, 178, 181 Runge–Kutta, sixth order, 468–475 algorithm, 473-475 derivation, 468-473 MHD rarefaction fans, 196, 218 Saha equation, 304, 416, 417–418, 437 thermal de Broglie wavelength,  $\lambda_{\rm e}$ , 418 scale height, L, 289, 290, 304 secant root finder, 462–466 multivariate, 205, 210-211, 465-466 univariate, 92, 93, 205, 364, 462-465 FORTRAN77 listing, 463–465 shear layer, 245 shock tube, 33 general MHD, 149 Sod, 93 shock waves, HD, 37-42, 80 entropy condition, **38–40**, 87 general frame, 58 hypersonic limit, 37 lab frame, 40-42shock strength, 58variable jumps, 37 shock waves, MHD, 155–172, 181, 182 entropy condition, **156**, 158, 159, 163, 169-171, 181 Euler branch, 164–166, 167, 170–172, 180evolutionary condition, **159**, 163, 170, 172fast shocks, **158–161**, 164–172, 181 switch-on shocks, 157, 164-165, 166, 167, 170-172, 179, 180, 203  $i \rightarrow j$  designation scheme, **159**, 167, 168 intermediate shocks, **159–161**, 165–172, 181shock types, **159**, 166, 170, 171, 180 slow shocks, 158-161, 165-172, 179, 181 switch-off shocks, **167**, 167 strength, 157, 207 variable jumps, 163, 207 slow point, 165, 169, 170, 180, 360, 375 slow speed,  $a_s$ , see magnetosonic speeds smooth flow, **22**, 72, 80, 82, 90 solar flares, 395, 398, 413 anomalous resistivity, 399

sound speed,  $c_s$ , 27, 127 adiabatic, 28 astrophysical values, 29 isothermal,  $c_{iso}$ , 28, 388 value in dry air at STP, 29 sound waves, 26-33 frequency,  $\omega$ , 28 linear algebra solution, 29–33 perturbation analysis, 26 secular equation, 31 solution to wave equation, 26-29wave vector,  $\vec{k}$ , 28 space-time diagrams, **74–76**, 77, 128 1-D MHD, 128, 202 Alfvén waves, 135, 137 event, 74 footprints, 76–78, 136 magnetosonic waves, 148 Riemann problem, 78–80, 90 sonic cone, 76 worldline, 74, 75 steady state definition, 34, 341 quasi, 329, 331 stellar winds (Weber–Davis), 361–366 additional assumptions, 361 asymptotic behaviour, 365–366, 376 boundary conditions, 357 profiles for  $\rho$ ,  $v_{\rm p}$ ,  $\psi$ , 364–365, 376 Stokes' theorem, 104, 446, 456 strain tensor, E, 316 strain components,  $\partial_j v_i$ , 314 streakline, 47 stream function,  $\psi$ , **258**, 259, 291 streaming motion, 9streamline, 47 streamtube, 48 stress tensor, T, 311-317components,  $T_{ij}$ , 311–317 cylindrical coordinates, 334–336 compressive stresses, 310, 312, 316 shear stresses, 310 trace, tr(T), 312–313, 337 relation to thermal pressure, p, 318 superfluids, 325 surface-conserved quantity, 104, 109, 110 synchrotron emission, 286 tangential discontinuity, HD, 35 tangential discontinuity, MHD, 36, 154 as limit to slow and Alfvén waves, 195 Theorem of hydrodynamics, **13–15**, 103, 117total energy equation, HD inviscid form, 17 scaled version, 337

viscid form, 320, 337

total energy equation, MHD, **109**, 110, 119 differenced 1-D steady state, 152, 178 resistive, 393, 432 steady state, 342, 372 two-fluid, non-ideal, 438 total energy, HD  $E_{\rm T}, 7, 15$ density,  $e_{\rm T}$ , 15 total energy, MHD density,  $e_{\rm T}^*$ , **108**, 393 triple umbilic, see wave families, MHD turbulence, 243-245, 253, 262, 283, 309, 324 - 326super-Alfvénic, 244 upwind; definition, 34 vector derivatives, 446-448 Cartesian coordinates, 447 cylindrical coordinates, 447 spherical polar coordinates, 447 vector identities, 443–445 with dyadics, 444-445vector potential,  $\vec{A}$ , **112**, 112–113, 118, 292, 293, 400-401, 456 contours as lines of  $\vec{B}$ , 292 evolution equation, 112 viscometer, 334, 336, 340 viscosity kinematic,  $\nu$ , **319**, 324, 393, 402 shear,  $\mu$ , **314**, 318, 319, 328 viscous dissipation, 321 viscous flow Couette, 334-336, 340 torque, 336, 340 forced between co-axial cylinders, 339 forced between plates, 328–329, 339 Hagen–Poiseuille, 333–334 open channel, 329-332 plane laminar, 327–328 volume-conserved quantity, 14, 18, 109, 110 vorticity,  $\vec{\omega}$ , 23, **105**, 254, 309, 319 comparison with  $\vec{B}$ , 24, 339 vorticity equation, 23 wave equation Alfvén waves, 131 sound waves, 27, 246

wave families, HD, 86, 130, 202
wave families, MHD, 130, 128–149, 202–204, 206
Alfvén waves, 129, 130–139, 175 compressional, 145, 149
linear algebra solution, 131–134, 175 properties, 131, 134, 146–149, 174 torsion, 177, 354, 368
wave equation, 131

compound wave, 160, 200, 203, 204 degeneracy, 145-149entropy wave, 129, 202 magnetosonic waves, 129, 139-146 fast vs. slow waves, 142-145linear algebra solution, 140–141 perturbation analysis, 139–140 properties, 141, 146–149, 174 triple umbilic, 148, 190, 191-195, 198, 200-204, 209 weakly ionised medium, 380, 383 interpretation of velocities, 384 Weber–Davis constants, 342–349, 372 angular speed,  $\Omega_0$ , 343–344, 369 mass load,  $\eta,\,345\text{--}346,\,369$ specific angular momentum, l, 347-349, 369, 371 steady-state axisymmetry, 342 work-kinetic theorem, 402

Zemplén's theorem, *see* shock waves; entropy condition ZEUS-3D, 63, **77**, 119, 160, 244, 251–252, 254–257, 262–265, 283–285, 366–371