PHYS 2335 REVIEW

PART 1. INTRODUCTION

I. INFINITE SERIES

1. Convergence tests

An *infinite series* has the form: $S_\infty = \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \ldots$ which may be tested for convergence as follows:

a) Cauchy ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \begin{cases} < 1, & \text{convergent;} \\ > 1, & \text{divergent;} \\ = 1, & \text{indeterminant.} \end{cases}$$

b) Gauss test: If

$$\frac{a_n}{a_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^2},$$

where $B(n)$ is a finite function of n for all n, then $h > 1 \Rightarrow$ convergent; $h \leq 1 \Rightarrow$ divergent.

Alternatively, if

$$\frac{a_n}{a_{n+1}} = \frac{n^2 + a_1 n + a_0}{n^2 + b_1 n + b_0},$$

then $a_1 > b_1 + 1 \Rightarrow$ convergent; $a_1 \leq b_1 + 1 \Rightarrow$ divergent.

c) Cauchy integral test: If $S = \sum_{n=i}^{\infty} a_n$ and $a_n = f(n)$ where $f(x)$ is a continuous, monotonically decreasing function in x over the range $x = i$ to $x = \infty$, then S converges so long as

$$\int_i^\infty f(x)\, dx$$

is finite. Otherwise, S diverges.

2. Algebra of Sums

Let

$$S = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j}.$$

If $i = m - n$ and $j = n$, then we have $i \geq 0 \Rightarrow n \leq m$, and thus $0 \leq m < \infty$ and $0 \leq n \leq m$. Therefore,

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{m} a_{m-n,n}.$$

II. SERIES EXPANSIONS

1. Taylor Series

Series expansion of a function about $x = x_0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0), \qquad \text{where} \qquad f^{(n)}(x_0) \equiv \left. \frac{d^n f(x)}{dx^n} \right|_{x=x_0}.$$

2. Maclaurin Series

Taylor series with $x_0 = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0).$$

3. Binomial Expansion

is the Maclaurin series for $f(x) = (1+x)^m$:

$$(1+x)^m = \begin{cases} 1 + mx + \dfrac{m(m-1)}{2!}x^2 + \dfrac{m(m-1)(m-2)}{3!}x^3 + \ldots & m \in \mathbb{R}, \\ \displaystyle\sum_{n=0}^{m} \frac{m!}{n!(m-n)!} x^n & m \in \mathbb{Z} \text{ and } m > 0 \end{cases}.$$

For $m \in \mathbb{Z}$ and $m > 0$, the binomial series is a finite series, and converges for all x. Otherwise, series converges only for $|x| < 1$.

III. SIMPLE ORDINARY DIFFERENTIAL EQUATIONS

1. First Order ODEs

General form of a first order ODE is:

$$y'(x) Q(x,y) + P(x,y) = 0.$$

If $P(x,y) = P(x)$ and $Q(x,y) = Q(y)$, the differential equation is *separable*, in which case the solution $y(x)$ can be found by direct integration:

$$\int_{y_0}^{y} Q(y)\, dy = -\int_{x_0}^{x} P(x)\, dx.$$

2. Linear, Second Order ODEs with Constant Coefficients

$$y''(x) + p\, y'(x) + q\, y(x) = r. \qquad (1)$$

Trial solution for the *homogeneous* equation ($r = 0$): $y_\pm(x) = e^{a_\pm x} \quad \Rightarrow$

$$a_\pm = \frac{-p \pm \sqrt{p^2 - 4q}}{2},$$

where $a_\pm$ can be real (exponential solutions), complex (combined exponential and sinusoidal solutions), or degenerate ($a_+ = a_- = a$). In the latter case, $y_2(x) = xe^{ax}$ is a second, linearly independent solution from e^{ax}. Thus, the general solution to the homogeneous ODE is

$$y_H(x) = \begin{cases} Ae^{a_+ x} + Be^{a_- x}, & a_+ \neq a_-; \\ Ae^{ax} + Bxe^{ax}, & a_+ = a_- = a, \end{cases}$$

where A and B are constants set by the boundary conditions. The solution to the inhomogeneous equation (1) is then $y(x) = y_H(x) + r/q$.

IV. VECTORS

1. Definition of a Vector

See Part 2, §I.3.

2. Linear Independence

Let $\vec{v}_1$, $\vec{v}_2$, …, $\vec{v}_n$ be n different vectors. Then, $\vec{v}_n$ is linearly independent of $\vec{v}_1$, $\vec{v}_2$, …, $\vec{v}_{n-1}$ if and only if there are no values of a_i such that

$$\vec{v}_n = \sum_{i=1}^{n-1} a_i \vec{v}_i.$$

Alternatively and equivalently, $\vec{v}_n$ is linearly independent of $\vec{v}_1$, $\vec{v}_2$, …, $\vec{v}_{n-1}$ if and only if

$$\sum_{i=1}^{n} b_i \vec{v}_i = 0 \quad \Rightarrow \quad b_i = 0, \ \forall \ i = 1, n.$$

3. Products of Vectors

$a\vec{A} = (aA_x,\, aA_y,\, aA_z)$ — multiplication by a scalar

$\vec{A} \cdot \vec{B} = AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$ — scalar product

$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y,\ A_z B_x - A_x B_z,\ A_x B_y - A_y B_x)$ — vector product

$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) =$
$\quad -\vec{A} \cdot (\vec{C} \times \vec{B}) = -\vec{B} \cdot (\vec{A} \times \vec{C}) = -\vec{C} \cdot (\vec{B} \times \vec{A})$ — triple product

$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$ — double vector product

PART 2. VECTOR CALCULUS

I. PARTIAL DERIVATIVES

1. Definition

Let $f(x, y, z)$ be a continuous function of three independent variables. Then

$$\frac{\partial f}{\partial x} = \lim_{x \to x_0} \frac{f(x, y, z) - f(x_0, y, z)}{x - x_0}, \quad etc.$$

2. Chain Rule

Let $g(x, y) = f(x'(x, y), y'(x, y))$. Then,

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x}; \qquad \frac{\partial g}{\partial y} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial y}.$$

3. Coordinate Transformations, and the Definition of a Vector

Consider a *linear* coordinate transformation which changes the coordinates from $(x_1,\, x_2,\, x_3)$ to $(x'_1,\, x'_2,\, x'_3)$. Then

$$x'_i = \sum_{j=1}^{3} \frac{\partial x'_i}{\partial x_j} x_j.$$

If $\vec{A} = (A_x,\, A_y,\, A_z)$ transforms like the coordinates, that is

$$A'_i = \sum_{j=1}^{3} \frac{\partial x'_i}{\partial x_j} A_j,$$

then $\vec{A}$ is a vector (*e.g.*, $\vec{r}$, $\vec{v}$, $\vec{F}$, *etc.*) If $\vec{B} = (B_x,\, B_y,\, B_z)$ transforms like

$$B'_i = \sum_{j=1}^{3} \frac{\partial x_j}{\partial x'_i} B_j,$$

then $\vec{B}$ is a *dual*-vector (*e.g.*, $\nabla\phi$).

II. THE NABLA OPERATOR

1. Gradient

Let $\phi(x, y, z)$ be a scalar function of the coordinates. Then the *gradient* of the function is given by:

$$\nabla\phi \equiv \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}.$$

We write $\nabla = (\partial_x,\, \partial_y,\, \partial_z)$, where $\partial_\xi \equiv \partial/\partial\xi$.

Identity: $\nabla\phi \cdot d\vec{r} = d\phi$ (chain rule). Thus, along a surface of constant ϕ, $\nabla\phi \cdot d\vec{r} = 0$ (the gradient is perpendicular to any vector lying on the surface and therefore to the surface itself).

2. Divergence

Let $\vec{v} = (v_x(x,y,z),\, v_y(x,y,z),\, v_z(x,y,z))$ be a vector function of the coordinates. Then, the divergence is defined as

$$\nabla \cdot \vec{v} = \partial_x v_x + \partial_y v_y + \partial_z v_z.$$

If $\nabla \cdot \vec{v} = 0$, $\vec{v}$ is *solenoidal.*

3. Curl

The curl of a vector function $\vec{v}$ is defined as

$$\nabla \times \vec{v} = (\partial_y v_z - \partial_z v_y,\, \partial_z v_x - \partial_x v_z,\, \partial_x v_y - \partial_y v_x).$$

If $\nabla \times \vec{v} = 0$, $\vec{v}$ is *irrotational.*

4. Identities

Let ϕ and θ be scalar functions of the coordinates, and let $\vec{A}$ and $\vec{B}$ be vector functions of the coordinates. Then,

i) $\nabla(\phi\theta) = \phi\nabla\theta + \theta\nabla\phi$

ii) $\nabla(\phi/\theta) = \dfrac{\theta\nabla\phi - \phi\nabla\theta}{\theta^2}$

iii) $\nabla(\phi + \theta) = \nabla\phi + \nabla\theta$

iv) $\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B})$

v) $\nabla\cdot(\phi\vec{A}) = \phi\nabla\cdot\vec{A} + \vec{A}\cdot\nabla\phi$

vi) $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$

vii) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

viii) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} - \vec{B}(\nabla\cdot\vec{A}) + \vec{A}(\nabla\cdot\vec{B})$

ix) $\nabla\times(\phi\vec{A}) = \phi\nabla\times\vec{A} + \nabla\phi\times\vec{A}$

x) $\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}$

5. Second Derivatives

i) $\nabla\cdot\nabla\phi = \nabla^2\phi$. ∇^2 is called the Laplacian operator.

ii) $\nabla\times(\nabla\phi) = 0 \;\;\Rightarrow\;\;$ if $\nabla\times\vec{F} = 0$, $\vec{F} = \nabla\phi$. $\vec{F}$ is a conservative force, and ϕ is the scalar potential.

iii) $\nabla\cdot(\nabla\times\vec{A}) = 0 \;\;\Rightarrow\;\;$ if $\nabla\cdot\vec{B} = 0$, $\vec{B} = \nabla\times\vec{A}$. $\vec{A}$ is called the *vector potential.*

iv) $\nabla \cdot (\nabla \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})$, where

$$\nabla \vec{A} = \begin{bmatrix} \partial_x A_x & \partial_x A_y & \partial_x A_z \\ \partial_y A_x & \partial_y A_y & \partial_y A_z \\ \partial_z A_x & \partial_z A_y & \partial_z A_z \end{bmatrix}.$$

III. VECTOR INTEGRATION

1. Integration of a gradient

If $\nabla\phi = (\partial_x \phi,\, \partial_y \phi,\, \partial_z \phi)$ is given, then

$$\phi = \begin{cases} \displaystyle\int \partial_x \phi \, dx = \phi_1 + f(y,z), \\ \displaystyle\int \partial_y \phi \, dy = \phi_2 + g(z,x), \\ \displaystyle\int \partial_z \phi \, dz = \phi_3 + h(x,y). \end{cases}$$

Equate the first integral to the second, then the second integral to the third to determine g. Alternatively, equating the second to the third, then the third to the first will allow you to solve for h. Finally, equating the third to the first, then the first to the second will allow you to solve for f. Regardless of how you do it, you should get the same result for ϕ.

2. Line Integrals

Open integrals: $\displaystyle\int_C \phi \, d\vec{r}, \qquad \int_C \vec{A} \cdot d\vec{r}, \qquad \int_C \vec{A} \times d\vec{r}.$

Closed integrals: $\displaystyle\oint_C \phi \, d\vec{r}, \qquad \oint_C \vec{A} \cdot d\vec{r}, \qquad \oint_C \vec{A} \times d\vec{r}.$

i) $\displaystyle\int_C \phi \, d\vec{r} = \hat{x} \int_C \phi \, dx + \hat{y} \int_C \phi \, dy + \hat{z} \int_C \phi \, dz.$

The path is specified by two functions, $f(x,y,z) = 0$ and $g(x,y,z) = 0$.

For the x-integral, use $f = 0$ and $g = 0$ to solve both y and z in terms of x, substitute into the integral, and evaluate at the limits.

For the y-integral, use $f = 0$ and $g = 0$ to solve both z and x in terms of y, substitute into the integral, and evaluate at the limits.

For the z-integral, use $f = 0$ and $g = 0$ to solve both x and y in terms of z, substitute into the integral, and evaluate at the limits.

Remember, $\int \phi \, d\vec{r}$ is a vector!

ii) $\displaystyle\int_C \vec{A} \cdot d\vec{r} = \int_C A_x \, dx + \int_C A_y \, dy + \int_C A_z \, dz$ which is a scalar.

Use the path $f = 0$ and $g = 0$ to evaluate each term, as above. If $\vec{A}$ is irrotational (conservative), then $\vec{A} = \nabla\phi$, and thus

$$\int_1^2 \vec{A}\cdot d\vec{r} = \int_1^2 \nabla\phi\cdot d\vec{r} = \int_1^2 d\phi = \phi|_1^2 = \phi(2) - \phi(1).$$

For a closed path, points 1 and 2 are the same, and thus $\phi(2) = \phi(1)$, and

$$\oint_C \vec{A}\cdot d\vec{r} = 0 \quad \text{for an irrotational vector } \vec{A} \text{ only.}$$

iii) $\displaystyle\int_C \vec{A}\times d\vec{r} = \hat{x}\int_C (A_y\,dz - A_z\,dy) + \hat{y}\int_C (A_z\,dx - A_x\,dz) + \hat{z}\int_C (A_x\,dy - A_y\,dz).$

3. Surface Integrals

Open integrals: $\displaystyle\int_S \phi\,d\vec{\sigma}, \qquad \int_S \vec{A}\cdot d\vec{\sigma}, \qquad \int_S \vec{A}\times d\vec{\sigma}.$

Closed integrals: $\displaystyle\oint_S \phi\,d\vec{\sigma}, \qquad \oint_S \vec{A}\cdot d\vec{\sigma}, \qquad \oint_S \vec{A}\times d\vec{\sigma}.$

4. Volume Integrals

$\displaystyle\int_V \phi\,dV,$

$\displaystyle\int_V \vec{A}\,dV = \hat{x}\int_V A_x\,dV + \hat{y}\int_V A_y\,dV + \hat{z}\int_V A_z\,dV,$

where in Cartesian coordinates, $dV = dx\,dy\,dz$, in cylindrical coordinates, $dV = r\,dr\,d\phi\,dz$, and in spherical polar coordinates, $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$.

IV. THEOREMS OF VECTOR INTEGRATION

1. Gauss' Theorem

$$\oint_S \vec{A}\cdot d\vec{\sigma} = \int_V \nabla\cdot\vec{A}\,dV,$$

$$\oint_S \phi\,d\vec{\sigma} = \int_V \nabla\phi\,dV,$$

$$\oint_S \vec{A}\times d\vec{\sigma} = -\int_V \nabla\times\vec{A}\,dV.$$

2. Green's Theorem

$$\oint_S u\nabla v\cdot d\vec{\sigma} = \int_V u\nabla^2 v\,dV + \int_V \nabla u\cdot\nabla v\,dV,$$

$$\oint_S (u\nabla v - v\nabla u)\cdot d\vec{\sigma} = \int_V (u\nabla^2 v - v\nabla^2 u)\,dV.$$

3. Stokes Theorem

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S \nabla \times \vec{A} \cdot d\vec{\sigma},$$

$$\oint_C \phi \, d\vec{l} = -\int_S \nabla \phi \times d\vec{\sigma},$$

$$\oint_C \vec{A} \times d\vec{l} = -\int_S (d\vec{\sigma} \times \nabla) \times \vec{A}.$$

PART 3. LINEAR ALGEBRA

I. VECTOR SPACES

1. Definition

A Vector Space $\mathcal{V}$ is a set of elements for which two operations $\oplus$ and $\odot$ are defined as follows:

Let A, B, $C \in \mathcal{V}$, and let α, $\beta \in I\!R$. Then:

1. $A \oplus B \in \mathcal{V}$ (closure);
2. $A \oplus B = B \oplus A$ (commutative);
3. $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ (associative);
4. $\forall\, A \in \mathcal{V}, \exists\,!\, Z \in \mathcal{V} \,|\, A \oplus Z = A$ (zero element);
5. $\forall\, A \in \mathcal{V}, \exists\,!\, B \in \mathcal{V} \,|\, A \oplus B = Z$ (negative element);
6. $\alpha \odot A \in \mathcal{V}$ (multiplication by a real);
7. $\alpha \odot (A \oplus B) = (\alpha \odot A) \oplus (\alpha \odot B)$ (distributive);
8. $(\alpha + \beta) \odot A = (\alpha \odot A) \oplus (\beta \odot A)$ (distributive);
9. $(\alpha\beta) \odot A = \alpha \odot (\beta \odot A)$ (associative);
10. $1 \odot A = A$ (identity).

2. Examples of Vector Spaces

i) $x \in I\!R$, the set of all real numbers with ordinary addition ($\oplus$) and multiplication ($\odot$).

ii) $\vec{v} \in I\!R^n$, the set of all n-dimensional vectors with vector addition ($\oplus$) and multiplication of a vector by a real number ($\odot$).

iii) The set of all functions, $f(x)$, with ordinary addition ($\oplus$) and multiplication by a real number ($\odot$). This vector space is formally infinite dimensional, and is known as a *Hilbert Space*.

iv) $A \in I\!R^{mn}$, the set of all m by n matrices, with matrix addition ($\oplus$) and multiplication of a matrix by a real number ($\odot$).

II. MATRICES

1. Definition as a Vector Space

An $m \times n$ matrix is a 2-D array of numbers with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Matrix addition ($\oplus$): Let $A, B, C \in I\!R^{mn}$. $A+B=C$ if and only if $c_{ij} = a_{ij} + b_{ij}$ where $1 \le i \le m$ and $1 \le j \le n$.

Multiplication by a scalar ($\odot$): Let $A, B \in I\!R^{mn}$, $\alpha \in I\!R$. $B = \alpha A$ if and only if $b_{ij} = \alpha a_{ij}$.

2. Matrix Multiplication

In addition to $\oplus$ and $\odot$, we define *matrix multiplication* as follows. Let $A \in I\!R^{mn}$, $B \in I\!R^{np}$, and $C \in I\!R^{mp}$. Then $C = AB$ if and only if

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Thus, c_{ik} is the "dot-product" of the ith row of A and the kth column of B.

Note that $A \in I\!R^{mn}$ and $B \in I\!R^{pq}$ can only be multiplied together if they are *compatible*, namely if $n = p$, in which case the result is an m by q matrix.

Note that matrix multiplication does not necessarily commute. That is, $AB \neq BA$ in general. Indeed, BA may not be compatible even if AB is.

3. Dirac (bra-ket) notation

Write $1 \times n$ matrices (*i.e.*, row vectors) as a "bra": $\langle \vec{v}\,|$. Thus, the ith row of matrix A is $\langle \vec{a}_i|$.

Write $n \times 1$ matrices (*i.e.*, column vectors) as a "ket": $|\vec{v}\rangle$. Thus, the kth column of matrix B is $|\vec{b}_k\rangle$.

If $C = AB$, we write in bra-ket notation:

$$c_{ik} = \langle \vec{a}_i | \vec{b}_k \rangle \equiv \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Note that $|\vec{b}_k\rangle\langle\vec{a}_i|$ is the multiplication of an n by 1 matrix with a 1 by n matrix and thus is a well-defined operation resulting in an n by n matrix:

$$|\vec{b}_k\rangle\langle\vec{a}_i| = \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix} \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} = \begin{bmatrix} b_{1k}a_{i1} & b_{1k}a_{i2} & \cdots & b_{1k}a_{in} \\ b_{2k}a_{i1} & b_{2k}a_{i2} & \cdots & b_{2k}a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ b_{nk}a_{i1} & b_{nk}a_{i2} & \cdots & b_{nk}a_{in} \end{bmatrix}.$$

4. Other Items

For all points following, let A, B, and $C \in I\!R^{nn}$, and let a_{ij} be the i-jth element of matrix A, *etc.*

i) The Commutator: Define

$$[A, B] \equiv AB - BA$$

as the *commutator* of A and B. The commutator is zero if and only if A and B commute.

ii) Diagonal Matrices: If $a_{ij} = 0 \, \forall \, i \neq j$, A is *diagonal*.

iii) Trace: The *trace* of a matrix is defined as the sum of all the diagonal elements:

$$\text{Tr}(A) \equiv \sum_{i=1}^{n} a_{ii}.$$

iv) Identity Matrix and the Kroneker Delta: I is a diagonal matrix with 1's down the diagonal; thus $IA = AI = A$. The matrix elements of the identity matrix is the *Kroneker Delta*, δ_{ij}, where by definition, $\delta_{ij} = 1$ if $i = j$, or 0 if $i \neq j$.

v) Matrix Inverse: If $AB = BA = I$, B is the *inverse* of A and we write $B = A^{-1}$. Not all matrices have inverses. Those that have inverses are called *invertible* or *non-singular* while those that don't have inverses are called *non-invertible* or *singular*. If $C = AB$ and A and B are both invertible, so is C, and

$$C^{-1} = (AB)^{-1} = B^{-1}A^{-1}.$$

vi) Matrix Transpose: $\tilde{A}$ is the *transpose* of A if $\tilde{a}_{ij} = a_{ji}$. If $C = AB$,

$$\tilde{C} = (\widetilde{AB}) = \tilde{B}\tilde{A}.$$

vii) Symmetric Matrices: A is *symmetric* if $\tilde{A} = A$, and *antisymmetric* if $\tilde{A} = -A$.

viii) Orthogonal Matrices: A is *orthogonal* if $A^{-1} = \tilde{A}$. If $C = AB$ and A, B are orthogonal, then

$$\tilde{C}C = (\widetilde{AB})(AB) = \tilde{B}\tilde{A}AB = \tilde{B}IB = \tilde{B}B = I.$$

Thus $\tilde{C} = C^{-1}$ and C is orthogonal too.

ix) Similar Matrices: A and B are similar if $\exists \, C \, | \, B = C^{-1}AC$.

x) Normal Matrices: A is *normal* if $[A, \tilde{A}] = 0$ (*i.e.*, A commutes with its transpose).

III. DETERMINANTS

1. Definition

The *determinant* of $A \in \mathbb{R}^{22}$ is defined by

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \equiv a_{11}a_{22} - a_{12}a_{21} \in \mathbb{R}.$$

For $A \in \mathbb{R}^{nn}$, we perform a *cofactor expansion*, namely

$$|A| = \sum_{j=1}^{n} a_{ij}\mathcal{C}_{ij} \quad \text{(for any } 1 \leq i \leq n) \;\; = \sum_{i=1}^{n} a_{ij}\mathcal{C}_{ij} \quad \text{(for any } 1 \leq j \leq n),$$

where the *cofactor* C_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix obtained by "striking out" the ith row and jth column and multiplying by $(-1)^{i+j}$.

2. Properties of Determinants

For all that follows, A, $B \in \mathbb{R}^{nn}$, a_{ij} is the i-jth element of A, *etc.*, and $\langle\vec{a}_i|$ is the ith row of A, *etc.*

i) If any two rows or columns of A are swapped, the sign of $|A|$ changes (but the magnitude stays the same).

ii) If any two rows or columns of A are the same, $|A| = 0$.

iii) If $\langle\vec{b}_i| = \langle\vec{a}_i|$, $\forall\, i \neq k$, and $\langle\vec{b}_k| = \alpha\langle\vec{a}_k|$, $\alpha \in \mathbb{R}$, then $|B| = \alpha|A|$.

iv) Suppose $\langle\vec{c}_i| = \langle\vec{b}_i| = \langle\vec{a}_i|$, $\forall\, i \neq k$, and suppose further that $\langle\vec{c}_k| = \langle\vec{a}_k| + \langle\vec{b}_k|$. Then $|C| = |A| + |B|$.

v) If any row (column) of matrix A is a linear combination of any other rows (columns) of A, $|A| = 0$.

vi) The value of a determinant is unchanged if a multiple of one row (column) is added to another.

vii) The Product Theorem for Determinants: $|AB| = |A||B|$.

viii) A is non-invertible if and only if $|A| = 0$.

IV. SYSTEMS OF LINEAR EQUATIONS (GAUSS-JORDAN ELIMINATION)

1. Matrix Representation of a System of Linear Equations

Consider a system of three equations in three unknowns (x_1, x_2, x_3):

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1;$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2;$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3,$$

which can be written in a fashion consistent with matrix multiplication:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

which in turn can be wrtitten in compact matrix notation:

$$A|\vec{x}\rangle = |\vec{b}\rangle \qquad (2).$$

If $|\vec{b}\rangle = 0$, equations (2) are said to be *homogeneous*. Otherwise, equations (2) are *inhomogeneous*.

2. Types of Systems of Linear Equations

i) homogeneous, A invertible: the *trivial solution*, $|\vec{x}\rangle = 0$, is the unique solution.

ii) inhomogeneous, A invertible: $|\vec{x}\rangle = A^{-1}|\vec{b}\rangle$ is the unique solution. Formally, if x_i and b_i are the ith components of $|\vec{x}\rangle$ and $|\vec{b}\rangle$ respectively,

$$x_i = \frac{1}{|A|}\sum_{j=1}^{n} b_j \mathcal{C}_{ji},$$

where $\mathcal{C}_{ji}$ is the j-ith cofactor of A. Further, the i-jth element of A^{-1} is given formally by:

$$a_{ij}^{-1} = \frac{\mathcal{C}_{ji}}{|A|}.$$

iii) homogeneous, A non-invertible: the *trivial solution*, $|\vec{x}\rangle = 0$, is a solution, but an infinity of non-trivial ($|\vec{x}\rangle \neq 0$), may also exist (see §V EIGEN-ALGEBRA).

iv) inhomogeneous, A non-invertible: There are no solutions to equations (2) because in this case, the equations are inconsistent. For example, if two of the equations are

$$x_1 + x_2 + x_3 = 1 \qquad \text{and} \qquad x_1 + x_2 + x_3 = 2,$$

these can't be true simultaneously. The resulting matrix A has two identical lines, thus $|A| = 0$ and A is non-invertible.

3. Gauss-Jordan Elimination for Inhomogeneous, Invertible Systems of Equations

Consider the system of inhomogeneous equations:

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &= -2; \\ x_1 - 2x_2 + x_3 &= 0; \\ -2x_1 + x_2 - 2x_3 &= 3, \end{aligned}$$

written as

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & -2 & 1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}.$$

Gauss-Jordan elimination of this system of equations proceeds as follows:

$$\begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \left[\begin{array}{ccc|c} 3 & -1 & 2 & -2 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & -2 & 3 \end{array}\right] \qquad \begin{array}{ll} & \\ \text{step 1} & (2) - \frac{1}{3}(1) \to (2) \\ \text{step 2} & (3) + \frac{2}{3}(1) \to (3) \end{array}$$

$$\begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \left[\begin{array}{ccc|c} 3 & -1 & 2 & -2 \\ 0 & -\frac{5}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{2}{3} & \frac{5}{3} \end{array}\right] \qquad \begin{array}{ll} \text{step 3} & (1) - \frac{3}{5}(2) \to (1) \\ & \\ \text{step 4} & (3) + \frac{1}{5}(2) \to (3) \end{array}$$

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \quad \left[\begin{array}{ccc|c} 3 & 0 & \frac{9}{5} & -\frac{12}{5} \\ 0 & -\frac{5}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & -\frac{3}{5} & \frac{9}{5} \end{array}\right] \qquad \begin{array}{ll} \text{step 5} & (1) + 3(3) \to (1) \\ \text{step 6} & (2) + \frac{5}{9}(3) \to (2) \\ & \end{array}$$

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \quad \left[\begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & -\frac{5}{3} & 0 & \frac{5}{3} \\ 0 & 0 & -\frac{3}{5} & \frac{9}{5} \end{array}\right] \qquad \text{Normalise :} \quad \begin{array}{r} \frac{1}{3}(1) \to (1) \\ -\frac{3}{5}(2) \to (2) \\ -\frac{5}{3}(3) \to (3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

The solution to the system of equations is thus $\langle\vec{x}| = (x_1, x_2, x_3) = (1, -1, -3)$, which should be verified by substituting back into the original equations.

4. Gauss-Jordan Matrix Inversion

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 3 & 0 & -4 \\ -1 & -2 & 2 \end{bmatrix}.$$

First, verify that $|A| \neq 0$ to make sure A^{-1} exists. In this case, $|A| = -6$. Gauss-Jordan elimination to find A^{-1} then proceeds as follows:

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \quad \left[\begin{array}{ccc|ccc} 2 & 1 & -2 & 1 & 0 & 0 \\ 3 & 0 & -4 & 0 & 1 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{ll} & \\ \text{step 1} & (2) - \frac{3}{2}(1) \to (2) \\ \text{step 2} & (3) + \frac{1}{2}(1) \to (3) \end{array}$$

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \quad \left[\begin{array}{ccc|ccc} 2 & 1 & -2 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} & 0 & 1 \end{array}\right] \qquad \begin{array}{ll} \text{step 3} & (1) + \frac{2}{3}(2) \to (1) \\ & \\ \text{step 4} & (3) - \quad (2) \to (3) \end{array}$$

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \quad \left[\begin{array}{ccc|ccc} 2 & 0 & -\frac{8}{3} & 0 & \frac{2}{3} & 0 \\ 0 & -\frac{3}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 2 & 2 & -1 & 1 \end{array}\right] \qquad \begin{array}{ll} \text{step 5} & (1) + \frac{4}{3}(3) \to (1) \\ \text{step 6} & (2) + \frac{1}{2}(3) \to (2) \\ & \end{array}$$

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \quad \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{8}{3} & -\frac{2}{3} & \frac{4}{3} \\ 0 & -\frac{3}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 2 & 2 & -1 & 1 \end{array}\right] \qquad \text{Normalise :} \quad \begin{array}{r} \frac{1}{2}(1) \to (1) \\ -\frac{2}{3}(2) \to (2) \\ \frac{1}{2}(3) \to (3) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{4}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \quad \Rightarrow \quad A^{-1} = \frac{1}{6}\begin{bmatrix} 8 & -2 & 4 \\ 2 & -2 & -2 \\ 6 & -3 & 3 \end{bmatrix}.$$

One must always check after that $AA^{-1} = I$.

V. EIGEN-ALGEBRA

1. Eigenkets and eigenvalues

Suppose $A \in I\!R^{nn}$. If

$$A|\vec{x}\rangle = \lambda|\vec{x}\rangle, \tag{3}$$

then λ are the *eigenvalues* of A while $|\vec{x}\rangle$ are the *eigenkets* or *eigenvectors* of A.

For $A \in I\!R^{nn}$, there are as many as n different eigenvalues associated with A, and for each eigenvalue, there is a non-trivial eigenket. To find the eigenkets, rearrange (3) to get:

$$(A - \lambda I)|\vec{x}\rangle = 0, \tag{4}$$

and force $(A - \lambda I)$ to be singular (so the eigenkets are non-trivial) by setting:

$$|A - \lambda I| = 0. \tag{5}$$

Equation (5) (the *secular equation*) is an nth order polynomial whose roots are the n eigenvalues. For each eigenvalue, solve the system of equations (4) for the eigenket, $|\vec{x}\rangle$.

For a non-degenerate eigenvalue, one of the n equations will be redundant, and the eigenket can be found to within a constant.

For an i-fold degenerate eigenvalue (i roots of the secular equation (5) are the same eigenvalue), i equations will be redundant, leaving the eigenket known to within i free parameters, which can be broken up into i linearly independent eigenkets, each associated with the same eigenvalue.

2. Theorems

i) If $A \in I\!R^{nn}$ is a normal matrix, A and $\tilde{A}$ have the same eigenkets and eigenvalues. That is to say, if $[A, \tilde{A}] = 0$ and $A|\vec{x}\rangle = \lambda|\vec{x}\rangle$, then $\tilde{A}|\vec{x}\rangle = \lambda|\vec{x}\rangle$.

ii) If $A \in I\!R^{nn}$ is a normal matrix, the eigenkets of non-degenerate eigenvalues are orthogonal. That is to say, if $[A, \tilde{A}] = 0$ and $|\vec{x}_i\rangle$ and $|\vec{x}_j\rangle$ are the eigenkets associated with eigenvalues $\lambda_i \neq \lambda_j$ respectively, then $\langle\vec{x}_i|\vec{x}_j\rangle = 0$.

3. Normal Modes of Oscillation

Three masses are connected in a straight line with two springs of spring constant k. Let the centre mass be M and the two outside masses be m. Find the *normal modes* of oscillation.

Let x_j, $j = 1, 2, 3$, be the displacements in the x-direction of the three masses at any given time. Application of Newton's Second Law to the three masses yield the following equations:

$$-kx_1 + \ kx_2 \qquad = m\frac{d^2x_1}{dt^2}; \tag{6}$$

$$kx_1 - 2kx_2 + kx_3 = M\frac{d^2x_2}{dt^2}; \tag{7}$$

$$kx_2 - kx_3 = m\frac{d^2x_3}{dt^2}. \tag{8}$$

Seeking the normal modes means seeking harmonic (single frequency, ω) solutions of the type

$$x_j(t) = x_{0j}e^{i\omega t}. \tag{9}$$

Substituting (9) into (6), (7), and (8) and writing the equations in matrix notation yields:

$$\begin{bmatrix} \frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{M} & 2\frac{k}{M} & -\frac{k}{M} \\ 0 & -\frac{k}{m} & \frac{k}{m} \end{bmatrix} |\vec{x}\rangle = \omega^2 |\vec{x}\rangle,$$

where $\langle\vec{x}| = (x_1, x_2, x_3)$. To find the eigenvalues, ω^2, solve the secular equation [equation (5)]:

$$\begin{vmatrix} \frac{k}{m} - \omega^2 & -\frac{k}{m} & 0 \\ -\frac{k}{M} & 2\frac{k}{M} - \omega^2 & -\frac{k}{M} \\ 0 & -\frac{k}{m} & \frac{k}{m} - \omega^2 \end{vmatrix} = 0,$$

whose roots are

$$\omega^2 = 0, \quad \frac{k}{m}, \quad \frac{2k}{M} + \frac{k}{m}.$$

i) $\omega^2 = 0$. Solve equation (4):

$$\begin{bmatrix} \frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{M} & 2\frac{k}{M} & -\frac{k}{M} \\ 0 & -\frac{k}{m} & \frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,$$

to get three equations in three unknowns, namely

$$x_1 - x_2 = 0; \qquad x_1 - 2x_2 + x_3 = 0; \qquad -x_2 + x_3 = 0.$$

Note that the second equation is the sum of the other two, and thus is not independent. Together, the first and third equations imply $x_1 = x_2 = x_3 = a_1$, where a_1 is some parameter. Thus, the eigenket associated with the eigenvalue $\omega^2 = 0$ is

$$\langle\vec{x}| = a_1(1,1,1) = \frac{1}{\sqrt{3}}(1,1,1),$$

where $a_1 = 1/\sqrt{3}$ normalises the eigenbra.

ii) $\omega^2 = \frac{k}{m}$. Solve equation (4):

$$\begin{bmatrix} 0 & -\frac{k}{m} & 0 \\ -\frac{k}{M} & 2\frac{k}{M} - \frac{k}{m} & -\frac{k}{M} \\ 0 & -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,$$

to get

$$\langle\vec{x}| = a_2(1,0,-1) = \frac{1}{\sqrt{2}}(1,0,-1),$$

where $a_2 = 1/\sqrt{2}$ normalises the eigenbra.

iii) $\omega^2 = 2\frac{k}{M} + \frac{k}{m}$. Solve equation (4):

$$\begin{bmatrix} -2\frac{k}{M} & -\frac{k}{m} & 0 \\ -\frac{k}{M} & -\frac{k}{m} & -\frac{k}{M} \\ 0 & -\frac{k}{m} & -2\frac{k}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,$$

to get

$$\langle\vec{x}| = a_3(1, -2\tfrac{m}{M}, 1) = \frac{1}{\sqrt{2 + 4m^2/M^2}}(1, -2\tfrac{m}{M}, 1),$$

where a_3 is chosen to normalise the eigenbra. For $m = M/2$, $\langle\vec{x}| = \frac{1}{\sqrt{3}}(1,-1,1)$.