# ASSIGNMENT 3, PHYS 2335

## Assigned: Tuesday, October 3, 2006
## Due: Tuesday, October 10, 2006

1. Using GNUPLOT on the departmental UNIX system, plot and attach a hard copy of the following:

a) $f(x) = \sin x$ in the domain $[-\pi,\ \pi]$. Specify the range to be $[-1.1,\ 1.1]$.

b) $f(x, y) = \sin(xy)$ with $x \in [-2, 2]$ and $y \in [-2, 2]$.

c) $f(x) = x^2$ in the domain $[-2,\ 2]$. Use whatever range you think looks best.

d) $f(x, y) = x^2 + y^2$, with the same domain as part b).

2. In class, we determined that the equation of motion for a damped harmonic oscillator moving vertically near the surface of the Earth is given by:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = mg,$$

where $x(t)$ is the position of the mass $m$ at time $t$, $b$ is the damping constant (damping force $= bv$), $k$ is the usual spring constant, and $g$ is the acceleration of gravity.

a) Assuming $x(0) = 0$ and $v(0) = 0$, show that $x(t)$ is given by:

$$x(t) = \frac{g}{{\omega_0}^2}\left(1 - e^{-\omega t}\left[\frac{\omega}{\sqrt{\omega^2 - {\omega_0}^2}}\sinh\left(\sqrt{\omega^2 - {\omega_0}^2}\,t\right) + \cosh\left(\sqrt{\omega^2 - {\omega_0}^2}\,t\right)\right]\right), \qquad (1)$$

where $\omega = b/2m$, $\omega_0 = \sqrt{k/m}$, and where

$$\sinh\alpha \equiv \frac{e^{\alpha} - e^{-\alpha}}{2}; \qquad \cosh\alpha \equiv \frac{e^{\alpha} + e^{-\alpha}}{2}$$

(pronounced *sinch* and *caush*) are the hyperbolic sine and cosine functions.

b) Let $\Omega = \sqrt{|\omega^2 - {\omega_0}^2|}$ and thus:

$$\sqrt{\omega^2 - {\omega_0}^2} = \begin{cases} \Omega, & \omega \ge \omega_0; \\ i\Omega, & \omega < \omega_0, \end{cases}$$

where $i \equiv \sqrt{-1}$. Show that equation (1) can be written as follows:

$$x(t) = \begin{cases} 0, & \omega \to \infty, \text{ stiff}; \\ \dfrac{g}{{\omega_0}^2}\left[1 - e^{-\omega t}\left(\dfrac{\omega}{\Omega}\sinh\Omega t + \cosh\Omega t\right)\right], & \omega > \omega_0, \text{ overdamped}; \\ \dfrac{g}{{\omega_0}^2}\left[1 - e^{-\omega_0 t}(1 + \omega_0 t)\right], & \omega = \omega_0, \text{ critically damped}; \\ \dfrac{g}{{\omega_0}^2}\left[1 - e^{-\omega t}\left(\dfrac{\omega}{\Omega}\sin\Omega t + \cos\Omega t\right)\right], & \omega < \omega_0, \text{ underdamped}; \\ \dfrac{g}{{\omega_0}^2}(1 - \cos\omega_0 t), & \omega \to 0, \text{ undamped}. \end{cases} \qquad (2)$$

over...

where a Maclaurin series expansion will verify the following identities:

$$\sin\alpha = -i\sinh(i\,\alpha); \qquad \cos\alpha = \cosh(i\,\alpha).$$

c) Let $\omega_0 = 1$ and $g = 9.8$. Use GNUPLOT on the departmental UNIX system to plot equation (2) for four different values of $\omega$, namely $\omega = 2,\ 1,\ 0.5$, and 0, using a domain of $[0, 12.6]$ and a range of $[-0.5, 20]$ all on the same plot. Attach a hard copy of your plot, and label each graph (by hand, if you like) with its value of $\omega$ and whether it represents the overdamped, critically damped, underdamped, or undamped case.

d) Explain the designations *overdamped*, *critically damped*, and *underdamped*.

3. a) Find a unit vector perpendicular to the surface $x^2 + y^2 + z^2 = 3$ at the point (1,1,1).

b) Derive the equation of the plane tangent to the surface at (1,1,1).

HUGE HINT: If $\vec{A} = (A_x, A_y, A_z)$ is a vector perpendicular to a plane, the equation of that plane is given by

$$A_x x + A_y y + A_z z = D$$

where $D$ is a constant and evaluated by requiring that a given point lie on the plane.

4. Consider the rotation of coordinates studied in class, where the $(x', y')$ system was obtained by rotating the $(x, y)$ system through an angle $+\phi$ (*i.e.*, counterclockwise). Thus:

$$x' = x\cos\phi + y\sin\phi, \qquad y' = -x\sin\phi + y\cos\phi,$$

$$\frac{\partial x'}{\partial x} = \cos\phi, \qquad \frac{\partial x'}{\partial y} = \sin\phi, \qquad \frac{\partial y'}{\partial x} = -\sin\phi, \qquad \frac{\partial y'}{\partial y} = \cos\phi.$$

a) Now suppose we start with the $(x', y')$ system and rotate the axes through an angle $-\phi$ (clockwise). In this case, we arrive back at the $(x, y)$ system. Find expressions for this *reverse* transformation; namely, determine $x$ and $y$ in terms of $x'$ and $y'$.

b) From these relations, evaluate

$$\frac{\partial x}{\partial x'} \qquad \frac{\partial x}{\partial y'} \qquad \frac{\partial y}{\partial x'} \qquad \frac{\partial y}{\partial y'}$$

These should be the same as their "inverses" (where the "inverse" of $\partial y/\partial x'$ is $\partial x'/\partial y$, *etc.*), thus supporting the claim made in class that for rotations, there is no distinction between "vectors" and "dual-vectors".

5. a) For the following functions

$$f(x, y) = xe^y; \qquad g(x, y, z) = \sin[a(x^2 + y^2 + z^2)]; \quad a = \text{constant}$$

evaluate the two partial derivatives of $f(x, y)$ (namely $\partial f/\partial x$ and $\partial f/\partial y$) and the three partial derivatives of $g(x, y, z)$.

b) Now consider $h(u, v) = f(x(u, v), y(u, v))$ where $f(x, y)$ is the same function as in part a), $x(u, v) = uv$, and $y(u, v) = \ln(uv)$. Using the chain rule, evaluate $\partial h/\partial u$ and $\partial h/\partial v$ in terms of $u$ and $v$.