Thus for a fixed observer, who sees a flow moving with speed u, they will measure a density

$$\rho = \gamma^2 \rho_0$$

We thus interpret T^{00} as the relativistic energy density of the matter field, as the only energy contribution to the field is from the motion of the fluid (since it doesn't have any pressure).

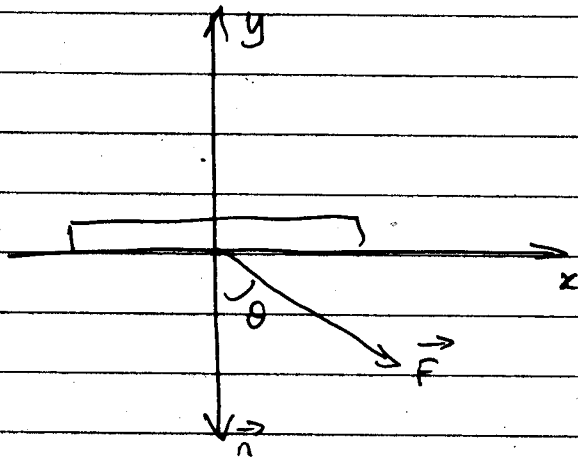
In matrix form T^{ab} can be written

$$T^{ab} = \rho \begin{bmatrix} 1 & u_x & u_y & u_z \\ u_x & u_x^2 & u_x u_y & u_x u_z \\ u_y & u_x u_y & u_y^2 & u_y u_z \\ u_z & u_x u_z & u_y u_z & u_z^2 \end{bmatrix}$$

Where we have used $\rho = \gamma^2 \rho_0$ & $u^\alpha \equiv \gamma(1, u^i)$

If you are familiar with stress calculations the lower right 3×3 matrix probably looks familiar, and has similarities to a stress tensor.

Consider the following problem:



Suppose we push a puck into a surface with force $\vec{F}$, where $\vec{F}$ is at an angle θ relative to the surface normal $\vec{n}$. Furthermore, suppose the area of the puck is A, then in terms of the classical stress tensor,

$$F^i = T^{ij} n_j A$$

So in this case

$$F^x = F\sin\theta = T^{xj} n_j A = -T^{xy} A$$
$$F^y = -F\cos\theta = T^{yj} n_j A = -T^{yy} A$$

Hence

$$T^{xy} = -\left(\frac{F}{A}\right)\sin\theta$$

$$T^{yy} = \left(\frac{F}{A}\right)\cos\theta.$$

Notice how the cross terms arise.

A more detailed analysis of the energy-momentum tensor (see Hartle p. 477) shows that T^{ab} can be broken into the following components

$$T^{ab} = \left(\begin{array}{c|c} \text{energy density} & \text{energy flux} \\ \hline \text{mom. density} & \text{stress tensor} \end{array} \right)$$

although since $T^{ab} = T^{ba}$ there is really no distinction between energy flux & momentum density.

Given this form for T^{ab} we next show that the equation of motion for the dust can be written

$$\partial_b T^{ab} = 0$$

From the matrix form of T^{ab} the time component of the equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

which can be written

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \vec{u}) = 0$$

which is the continuity equation (equivalently mass conservation).

Since matter & energy are equivalent in SR, the conservation of energy is thus equivalent to $\partial_b T^{0b} = 0$

We can calculate the $\partial_b T^{ib}$ components as well

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \frac{\partial}{\partial x}(\rho u_x \vec{u}) + \frac{\partial}{\partial y}(\rho u_y \vec{u}) + \frac{\partial}{\partial z}(\rho u_z \vec{u}) = 0$$

Which can be re-writte in a more compact form:

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (u . \nabla) \vec{u} \right] = 0$$

which is the Euler equation for a pressureless fluid.

Thus we've show $\partial_b T^{ab} = 0$. Note to extend this energy-momentum conservation equation to GR we apply the Principle of Minimal Gravitational Coupling:

$$\partial_b T^{ab} \longrightarrow \nabla_b T^{ab} = 0$$

What about pressure? In the limit $p \to 0$ we must recover $\rho_0 u^a u^b$. So it makes sense to add a term linear in p:

$$T^{ab} = \rho_0 u^a u^b + p S^{ab}$$

where S^{ab} must be symmetric & rank 2.

What choices do we have for rank 2 tensors? Firstly, $u^a u^b$, but we also have the metric g^{ab}. Hence the simplest form we can consider is

$$S^{ab} = \lambda u^a u^b + \mu g^{ab}$$

where λ, μ are constants.

We can then proceed in the same manner as earlier & examine $\partial_b T^{0b}$ & $\partial_b T^{ib}$ assuming a S.R. system.

In this case the limiting equations will be the full Euler equation

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right] = -\nabla p$$

as well as the continuity equation. Equating the Euler equation to that derived using S^{ab}, we find that $\lambda = 1, \mu = -1$.

Hence the perfect fluid (ie one fully characterized by its rest frame density & pressure) energy-momentum tensor is

$$T^{ab} = (\rho_0 + p) u^a u^b - p g^{ab}$$

There will also be an equation of state, so that $p \equiv p(\rho, T)$. Usually in GR we work in situations where $T \simeq$ constant so that $p \equiv p(\rho)$. For example, in cosmology, we use

$$p \equiv p(\rho) = w\rho$$

where w is related to the adiabatic index by $w = \gamma - 1$. There are two cases of particular interest, $w=0$ (dust) and $w=1/3$, which corresponds to radiation or a relativistic fluid (as in the early universe).

The Full Field Equations

With the energy momentum tensor as a source of curvature the field equations are

$$G_{ab} = \kappa T_{ab}$$

where κ is a coupling constant. Under the slow motion approximation we recover the Poisson equation in the weak field limit (see section 12.10). This result gives

$$\kappa = \frac{8\pi G}{c^4}$$

In actual fact because the Bianchi identities ~~satisfy~~ yield $\nabla_b G^{ab}=0$, and $\nabla_b T^{ab}=0$ we can add another term in g^{ab} to the field equations because $\nabla_b g^{ab}=0$. This is the cosmological constant term, Λ:

$$G_{ab} - \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

which is the full field equation of General Relativity.

Relativistic Cosmology

Before we begin to study cosmology we must ask a few philosophical questions about whether cosmology is really a science or not. Since the Universe is (to all intents and purposes) unique, we cannot run any experiments on it or create new ones, or change it in any significant way. Further, we occupy only a miniscule fraction of it and the information we have gathered comes solely from photons (at least for the moment). We have no direct physical measurements of distances (for example).

Nonetheless, we can only proceed by assuming that the Laws of Physics hold at all places, at all times. One might perhaps envisage a universe in which fundamental constants change in time or space (smoothly), however there is limited observational evidence for this. Equally importantly, we do not have direct measurements of gravitation (ie experimental tests) on very large scales - this is currently the subject of debate (look for literature on Modified Newtonian Dynamics or "MOND"). However, we shall proceed as with the mainstream, that we can apply the physical laws from our local patch of spacetime to all other parts of it.

Olber's Paradox

This is a very simple idea: if the universe is infinite in size & age, how is the night sky dark?

If we adopt a view of the universe as being filled with stars at an average density n_0 then the number of stars with in shell of thickness dr at a distance r is

$$dN = 4\pi r^2 n_0 \, dr$$

Clearly if we integrate from the origin to r the total number of stars is trivially,

$$N = \frac{4\pi r^3 n_0}{3}$$

The flux arriving at the origin from the shell of stars is $L/4\pi r^2$ where L is the total luminosity of the shell. Assuming each star has a luminosity L_* then the flux at the origin from each shell is

$$dF = \frac{4\pi r^2 (n_0 L_*) \, dr}{4\pi r^2}$$

If we integrate the total flux from a series of shells from 0 to r, then

$$F = (n_0 L_*) r$$

which clearly <u>diverges</u> as $r \longrightarrow \infty$.

Possible objections: Material between us & the distant stars either intercepts or scatters the light. <u>Doesn't work!</u> Eventually material reaches thermal equilibrium with the background radiation.

Resolution: Suppose the universe has a finite age (- which we are pretty sure of!) Then the limiting distance is bounded and so is the flux.

Note: This apparently trivial observation did not prevent people from believing in static universes (Einstein ~~clear~~ invoked the cosmological constant to create a static solution).

As an aside, we can also appeal to expansion of the universe to reduce the flux from distant shells. However, Charlier (1908) proposed an intriguing idea:

Suppose that the distribution of stars is such that the number in a shell goes as

$$dN \propto r^{D-1}\, dr$$

Where D is a constant. $D=3$ would correspond to a uniform density of stars like we just examined. The flux from a shell is thus

$$dF \propto r^{D-3}\, dr$$

and so the total flux is proportional to

$$F \propto \int_0^r r^{D-3}\,dr \propto r^{D-2}$$

So for a distribution of stars for which $D<2$ the flux converges even when $r \to \infty$. Intriguingly, this would correspond to a fractal like distribution of material and the distribution of galaxies on small scales does look similar to a fractal distribution. However, as we go to larger scales the distribution becomes completely uniform.

Key Principles of Cosmology

(1) The Cosmological Principle: The Universe is spatially isotropic & homogeneous.

While clearly not true locally, observations show that on the very largest scales the universe does appear to be isotropic (modulo small perturbations).

It should be noted that we cannot give any evidence for homogeneity - it is an assumption.

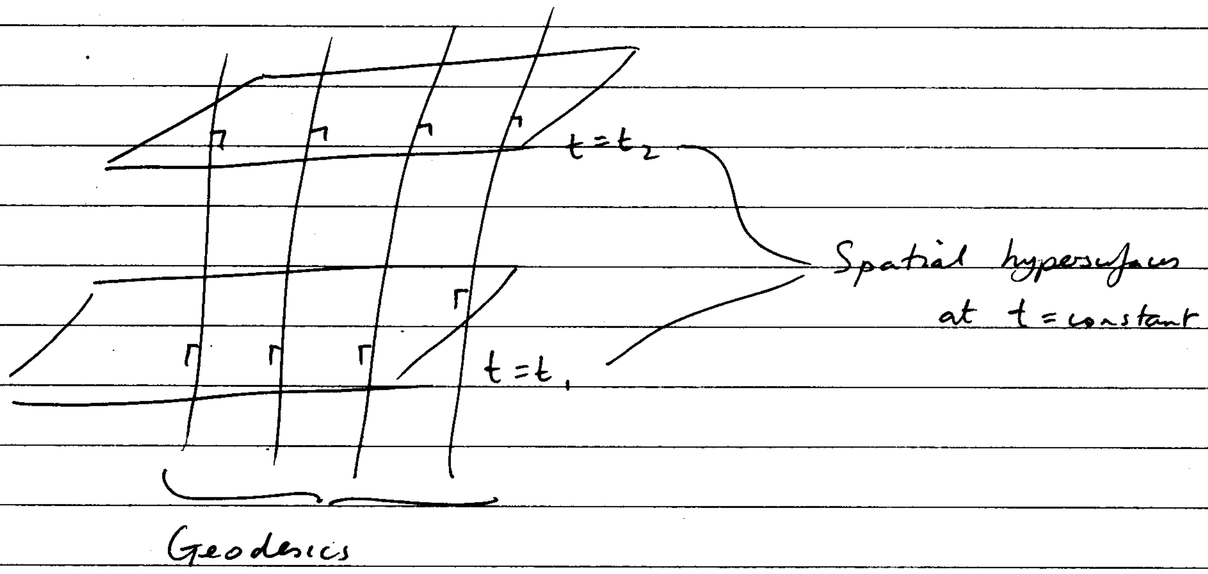
The Cosmological Principle is an extension of Copernican ideas that the Earth should not be special in any way. It extends the idea to be "our place in the universe" is not special.

Note: an extension of the Cosmological Principle is the "Perfect Cosmological Principle", that the universe is homogeneous & isotropic in both space & time.

This is clearly at odds with our modern understanding, but was the underlying idea behind the Steady State Theory. (Although see the idea of "Chaotic Inflation" for a resurrection of this idea.)

(2) Weyl's Postulate: The world lines of the cosmological fluid do not intersect, except at singular points in the past/future.

This statement means that the world lines (geodesics) are everywhere orthogonal to a family of spatial hypersurfaces of $t = \text{constant}$.



It is worth noting that galaxies don't actually obey this idea - we know they merge for example. However, the scale on which we need this concept to hold is far larger than that of an individual galaxy. Equivalently, you can think of "smearing out" the local motions of galaxies.

Friedmann - Lemaître - Robertson - Walker Line Element

This is the most general metric used in studies of global cosmological evolution. Before we begin a derivation of the line element in 4d, it is helpful to consider a 2-d + 1 time example that will be analogous to our later discussion.

For a series of spacelike slices (t = constant) a spatial metric that satisfies homogeneity & isotropy is

$$dl^2 = dx^2 + dy^2$$

ie the traditional Euclidean metric. We can rewrite this in circular polars to show isotropy & it is invariant under spatial displacements which implies homogeneity.

However, we could equally well have a 2d surface with curvature. An example would be the 2-sphere provided by the surface of a three dimensional sphere

$$x^2 + y^2 + z^2 = a^2$$

The metric for this surface is then

$$dl^2 = dx^2 + dy^2 + dz^2$$

although we must eliminate the dz^2 to have only 2 coordinates. Since $x^2+y^2+z^2 = a^2$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0$$

$$\therefore \quad dz^2 = \frac{(x\,dx + y\,dy)^2}{a^2 - x^2 - y^2}$$

Defining ϕ via $x = r\cos\phi$, $y = r\sin\phi$ we can rewrite dl^2 as

$$dl^2 = \frac{a^2\,dr^2}{a^2 - r^2} + r^2\,d\phi^2$$

and a rescaling $r \to r' = \frac{r}{a}$ gives

$$dl^2 = a^2\left(\frac{dr'^2}{1 - r'^2} + r'^2\,d\phi^2\right)$$

The extent of the 2-sphere is of course finite, it has area $4\pi a^2$. $r' = 0$ corresponds to the North Pole of the sphere, while the ~~coordinate~~ $r' = 1$ singularity is just a coordinate singularity. (This particular pair of coordinates only covers half the sphere.)

We now extend this argument to 4d. Firstly we must decide how the time coordinate should be separated. Weyl's Postulate addresses this point.

Suppose we place an observer on each world line of the fluid. Then they can all agree on an initial time t_0. If each observer has coordinates x, y, z then we introduce a coordinate system such that the x, y, z are constant, and the observers are at rest in this coordinate system. The orthogonality requirement of Weyl's Postulate means that we can write the line element as

$$ds^2 = dt^2 - h_{ij}\,dx^i dx^j = dt^2 - dl^2$$

Thus t plays the role of cosmic or world time.

For the 3-space part of the metric we now repeat our earlier 2-d analysis by embedding a 3-sphere into a 4-d space.
For a 3-sphere of radius a:

$$x^2 + y^2 + z^2 + w^2 = a^2 \quad \text{——— (A)}$$

hence we define dl^2 by

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2$$

and using the hypersurface equation (A) to eliminate w we get

$$dl^2 = dx^2 + dy^2 + dz^2 + \frac{(x\,dx + y\,dy + z\,dz)^2}{a^2 - x^2 - y^2 - z^2}$$

If we now introduce spherical polar coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

then

$$dl^2 = dr^2 + \frac{r^2 dr^2}{r^2 - a^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

$$\Rightarrow \quad dl^2 = \frac{dr^2}{1 - r^2/a^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

We can also multiply this spatial part by an arbitrary function of time (this will not change homogeneity or isotropy).

We can write for the full line element

$$ds^2 = dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

Where we have absorbed the length dimension into $R(t)$, which thus makes r dimensionless. Further, in this case $k \equiv a^{-2}$ and is also dimensionless.

$R(t)$ is called the scale factor, and can be viewed as roughly governing the size of the universe at a given time. The physical radial positions are now $x = Rr$, and r is said to be a comoving coordinate. The r coordinate can always be scaled so that we need only consider the cases $k = 1, 0, -1$. While our derivation of the line element assumed $k>0$ a more general argument (See D'Inverno 22.7) shows that the line element is also valid for $k=0$ & $k=-1$.

The values of k categorize the geometry of the cosmology as follows:

$k=1$ $\Rightarrow$ closed, curved spacetime
$k=0$ $\Rightarrow$ open, Euclidean spacetime
$k=-1$ $\Rightarrow$ open, curved spacetime.

We'll review why this is the case in the last lecture.

The Friedmann Equation(s)

Substituting the FLRW line-element into the Einstein Field Equations with $T^{ab} = (\frac{p}{c^2} + \rho)u^a u^b - p g^{ab}$ yields

$$8\pi G\rho = \frac{3(kc^2 + \dot{R}^2)}{R^2} - \Lambda \quad \text{———} (A)$$

$$\frac{8\pi G p}{c^2} = -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{kc^2}{R^2} + \Lambda \quad \text{———} (B)$$

Where dots denote differentiation wrt t. We can rearrange (A) to give

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3}$$

and substituting into (B) then gives

$$\frac{8\pi G p}{c^2} = -\frac{2\ddot{R}}{R} - \frac{8\pi G\rho}{3} + \frac{2\Lambda}{3}$$

$$\Rightarrow \quad \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}$$

If we multiply (A) by R^2 and differentiate wrt t, then we derive

$$\frac{\ddot{R}}{R} = \frac{4\pi G}{3}\left\{\dot{\rho}\frac{R}{\dot{R}} + 2\rho\right\} + \frac{\Lambda}{3}$$

These two equations for $\ddot{R}/R$ can be equated

$$\Rightarrow \quad \frac{4\pi G}{3}\left\{\dot{\rho}\frac{R}{\dot{R}} + 2\rho\right\} = -\frac{4\pi G}{3}\left\{\rho + \frac{3p}{c^2}\right\}$$

after cancelling the cosmological constant terms

We thus arrive at

$$\left\{ \dot{\rho}\frac{R}{\dot{R}} + 3\rho + \frac{3p}{c^2} \right\} = 0$$

Multiply by $R^2\dot{R}$

$$\Rightarrow \quad \dot{\rho}R^3 + 3R^2\dot{R}\rho + 3R^2\dot{R}\frac{p}{c^2} = 0$$

$$\therefore \quad \frac{d}{dt}\left\{\rho R^3\right\} + \frac{p}{c^2}\frac{d}{dt}\left\{R^3\right\} = 0 \quad \text{——— (C)}$$

This is the relativistic cosmology equivalent of

$$dE + p\,dV = 0$$

ie a cosmological analogue of the First Law of Thermodynamics.

This result can also be shown to follow from the conservation equation

$$\nabla_b T^{ab} = 0$$

This should not be a surprising result since the field equations satisfy the contracted Bianchi identity

$$\nabla_b\left(G^{ab} - \Lambda g^{ab}\right) = 0.$$

For the "late" universe (ie a universe filled with galaxies rather than radiation) we can set $p=0$ to a high degree of accuracy. In this case equation (B) gives

$$0 = -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{kc^2}{R^2} + \Lambda$$

which after multiplying by $R^2\dot{R}$, can be written

$$0 = 2R\dot{R}\ddot{R} + \dot{R}^3 + kc^2\dot{R} - \Lambda\dot{R}R^2$$

$$= \frac{d}{dt}\left\{ R(\dot{R} + kc^2) - \frac{\Lambda R^3}{3} \right\}$$

and we can integrate to give

$$R(\dot{R} + kc^2) - \frac{\Lambda R^3}{3} = C$$ where C = constant of integration

We can actually determine the exact value of C using equation (A). If we multiply by $R^3/3$ then we get

$$R(\dot{R}^2 + kc^2) - \frac{\Lambda R^3}{3} = \frac{8\pi\rho R^3}{3} \qquad \text{(D)}$$

hence $C = \frac{8\pi}{3}\rho R^3$. That C is a constant also follows from the conservation equation (C) when $p = 0$. Therefore, up to a numerical factor, we can equate C with the energy content of a volume V of the cosmological fluid.

Substituting for C on the RHS of (D) we can rearrange it to give

$$\dot{R}^2 = \frac{C}{R} + \frac{1}{3}\Lambda R^2 - kc^2$$

Which is called Friedmann's equation. It governs the evolution of the scale factor in terms of the energy content, cosmological constant and the spatial curvature k.

Null Geodesics in the FLRW Geometry

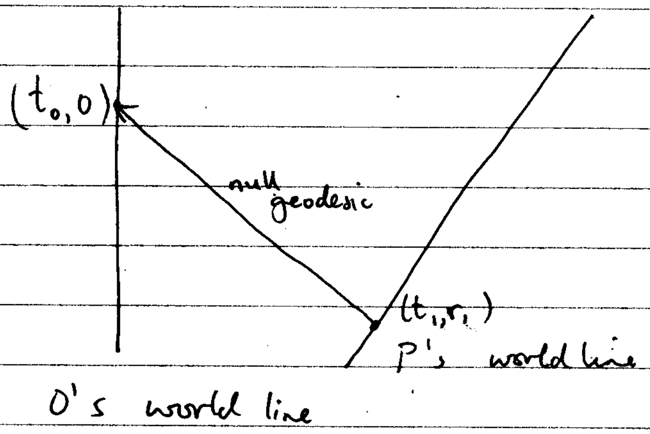
These geodesics are especially simple if we consider paths between two observers at $r = \theta = \phi = \text{constant}$. Using the metric geodesic in its form as the line element, we have

$$0 = c^2 dt^2 - R^2(t) \frac{dr^2}{1-kr^2}$$

$$\therefore \quad \frac{c\,dt}{R(t)} = \pm \frac{dr}{(1-kr^2)^{1/2}}$$

The +ve sign corresponds to an outgoing ray, while -ve sign corresponds to an incoming ray.

Let's consider two observers O & P:



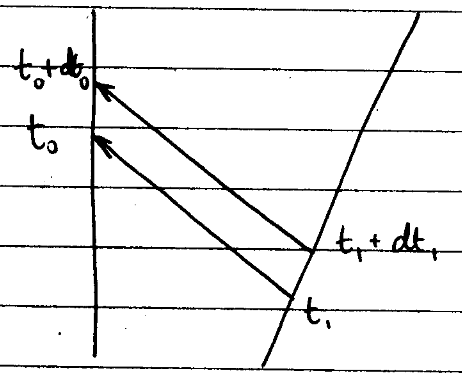
For the incoming ray observed by O

$$\int_{t_1}^{t_0} \frac{c\,dt}{R(t)} = -\int_{r_1}^{0} \frac{dr}{(1-kr^2)^{1/2}} = f(r_1)$$

where (exercise)

$$f(r_1) = \begin{cases} \sin^{-1} r_1 & \text{if } k=+1 \\ r_1 & \text{if } k=0 \\ \sinh^{-1} r_1 & \text{if } k=-1 \end{cases}$$

For two rays separated in time



Then since both r values are constant for the observers, we must have

$$\int_{t_1}^{t_0} \frac{c\,dt}{R(t)} = \int_{t_1+dt_1}^{t_0+dt_0} \frac{c\,dt}{R(t)}$$

Since both sides will equal $f(r_1)$. Thus

$$\int_{t_0}^{t_0+dt_0} \frac{c\,dt}{R(t)} = \int_{t_1}^{t_1+dt_1} \frac{c\,dt}{R(t)}$$

Assuming $R(t)$ changes little in the intervals dt_0 & dt_1 we get

$$\frac{c\,dt_0}{R(t_0)} = \frac{c\,dt_1}{R(t_1)}$$

If we recall that for the observers with $\dot{r} = \dot{\theta} = \dot{\phi} = 0$ their proper time is the same as coordinate time:

$$c^2 d\tau^2 = c^2 dt^2$$

Then for an expanding universe where $R(t_0) > R(t_1)$ we get

$$dt_0 = \frac{R(t_0)}{R(t_1)} dt_1$$

and hence $dt_0 > dt_1$.

Since frequency $\propto \frac{1}{dt}$

$$\frac{\nu_1}{\nu_0} = \frac{R(t_0)}{R(t_1)} = 1 + z = \frac{\lambda_0}{\lambda_1}$$

where z is the redshift. For a contracting universe $R(t_1) > R(t_0)$ and we would measure a blue shift.

For small separations $R(t_1) \simeq R(t_0 - dt)$ and a Taylor expansion gives

$$1 + z \simeq 1 + \frac{\dot{R}(t_0)}{R(t_0)} dt$$

We can substitute for $\frac{dt}{R(t_0)}$ using

$$\int_{t_1}^{t_0} \frac{c\,dt}{R(t)} \simeq \int_{t_1}^{t_1+dt_1} \frac{c\,dt}{R(t_1)} \simeq \frac{c\,dt}{R(t_1)} = \frac{c\,dt}{R(t_0 - dt)} \simeq \frac{c\,dt}{R(t_0)}$$

$$= f(r_1) \simeq r_1 \qquad \text{to get}$$

$$cz \simeq \dot{R}(t_0)\, r_1$$

so redshift is proportional to distance for small separations. Writing $d = Rr_1$ we recover Hubble's Law:

$$v = cz = \frac{\dot{R}(t_0)}{R(t_0)} R(t_0) r_1 = Hd \qquad \text{where } H = \frac{\dot{R}(t_0)}{R(t_0)}$$

Luminosity Distance

Distance measures in cosmology are very important. Further their derivations can be non-trivial. We'll look at one definition - the Luminosity distance, D_L.
Using the definition of flux as $F = L/4\pi r^2$, where L is the luminosity, in a static Euclidean space we can define the notion of a luminosity distance by

$$D_L^2 = \frac{L}{4\pi F}$$

For an expanding universe we must consider its effects on the measured flux.
Firstly, we have just shown that the frequency will fall by a factor of $1+z$ (assuming an expanding universe). For $t_1 < t_0$

$$\frac{E_0}{E_1} = \frac{h\nu_0}{h\nu_1} = \frac{R(t_1)}{R(t_0)} = \frac{1}{1+z}$$

This result follows for the energy of a single photon. However, in calculating the flux we must also take into account what happens to the unit time interval Δt_0 & hence the <u>number of photons received per unit time</u>.
The analysis is exactly as before and we find the total flux is reduced by a further redshift factor, making the total flux $\frac{1}{(1+z)^2}$ of what might be expected.
Interpret this as follows: individual photons are redshifted, and the receiving interval is time-dilated.

By the time the photons reach the observer, the size of the light sphere will be $R(t_0)r_1$. Hence the total flux received will be

$$F = \frac{L}{4\pi R(t_0)^2 r_1^2} \frac{1}{(1+z)^2}$$

and if we substitute back into the definition of D_L we find:

$$D_L^2 = R(t_0)^2 r_1^2 (1+z)^2$$

$$\Rightarrow \quad D_L = R(t_0) r_1 (1+z)$$

Note this is different to D'Inverno's definition but is consistent with that used in many cosmological texts. The issue is whether one includes a factor of $(1+z)^2$ in the definition of D_L^2.

So in this case the distance is explicitly dependent on both the scale factor and the redshift.

Another useful definition is the radial proper distance, the distance between the objects on the $t=$ const hypersurface:

$$D_p(t_0) = R(t_0) \int_0^r \frac{dr}{(1-kr^2)^{1/2}} = R(t_0) f(r) \simeq R(t_0) r \quad \text{for small } r$$

It should again be clear that saying the distance is $R(t_0)r$ is only an approximation when spatial curvature is considered.

Curvature & its effect on Geometry

We can get a feel for the overall geometry very quickly.
Recall that the proper distance was given by $f(r_1)R(t_0)$, where

$$f(r_1) = \begin{cases} \sin^{-1} r_1 & k=+1 \\ r_1 & k=0 \\ \sinh^{-1} r_1 & k=-1 \end{cases}$$

If we consider a coordinate transform to χ where

$$\chi = \begin{cases} \sin^{-1} r & k=+1 \\ r & k=0 \\ \sinh^{-1} r & k=-1 \end{cases}$$

Then we find the FLRW line element can be written

$$ds^2 = c^2 dt^2 - R^2(t)\left[d\chi^2 + S^2(\chi)\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right]$$

where

$$S(\chi) = \begin{cases} \sin \chi & k=+1 \\ \chi & k=0 \\ \sinh \chi & k=-1 \end{cases}$$

Let's consider the $k=+1$ case, so that $S(\chi) = \sin \chi$.
As noted earlier, the $t=$constant hypersurfaces allow us to define physical separations over the whole of the spacetime.

Hence, at t_0 the proper circumference of a circle centered on the origin with $\chi = $ constant and $\theta = \frac{\pi}{2}$ is $2\pi \sin\chi R(t_0)$. The proper radius is $\chi R(t_0)$. Thus the ratio is

$$\frac{2\pi \sin\chi}{\chi} < 2\pi \quad \forall \ \chi$$

The circumference reaches at maximum for $\chi = \frac{\pi}{2}$ & tends to zero again as $\chi \to \pi$.

The volume of a 3-sphere in this geometry is (ex)

$$V = \int_{\chi=0}^{\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (R(t_0)d\chi)(R(t_0)\sin\chi \, d\theta)(R(t_0)\sin\chi \sin\theta \, d\phi)$$

$$= 2\pi^2 R^3(t_0)$$

This is why the scale factor is often stated as governing the size of the universe.

Any radial geodesic in this geometry eventually returns to its starting point! This should not be a surprising result though, as when we constructed the line element we specifically used the geometry of a 3-sphere. Hence, we call this geometry closed.

The remaining cases $k=0,-1$ are all open since radial geodesics can never return to their origin.

Cosmological Models

Friedmann's equation

$$\dot{R}^2 = \frac{C}{R} + \frac{\Lambda R^2}{3} - kc^2$$

contains within it the solutions to all late-time cosmological models of interest, and some that are not interesting!
The values of the constants are constrained by

$$C > 0, \quad -\infty < \Lambda < \infty \qquad k = -1, 0, +1$$

The equation can be solved using elliptic functions, or under special cases (such as $\Lambda = 0$) using more elementary means. See D'Inverno for details. We shall just examine specific cases by plotting the behaviour of $R(t)$.

Case 1: $k = 0$ $\Lambda = 0$ The Einstein-de Sitter model

This is the simplest physically plausible model & was used extensively until ~ year 2000 when it became clear that there is a cosmological constant.

For this model it is trivial to solve

$$\dot{R}^2 = \frac{C}{R}$$

$$\Rightarrow \quad R^3(t_0) = \frac{9}{4} C t_0^2 \qquad \text{where } t_0 \text{ is the age of the Universe.}$$

The expansion factor grows as

$$R \propto t^{2/3} \quad \text{and} \quad \frac{\dot{R}}{R} = H = \frac{2}{3t}$$

Thus the Hubble ~~constant~~ parameter at t_0 is just $2/3t_0$, showing the simple relationship between the age of the universe & the Hubble constant in this model.

The deceleration parameter q is defined by

$$q(t) = -\frac{R\ddot{R}}{\dot{R}^2}$$

this term can be seen to play a role in the Friedmann equation if we differentiate it & multiply by $-R/2\dot{R}^3$; then we recover

$$-\frac{R\ddot{R}}{\dot{R}^2} = \frac{C}{2R\dot{R}^2} - \frac{\Lambda}{3}\frac{R^2}{\dot{R}^2}$$

$$\Rightarrow \quad q(t) = \frac{\Omega}{2} - \frac{\Lambda}{3H^2}$$

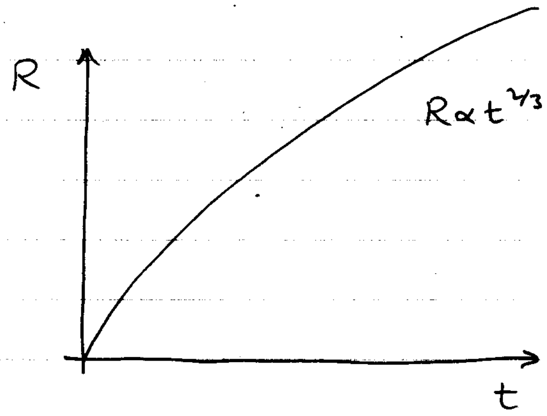
where $\Omega = \frac{8\pi G\rho}{3H^2}$ is called the density parameter.

We can quickly show that $q(t) = \frac{1}{2}$ for the Einstein-de Sitter model

$$\Rightarrow \quad \Omega = 1$$

Hence the density of the universe in this model is given by

$$\rho = \frac{3H^2}{8\pi G}.$$



Plot of $R(t)$ for EdS model.

Case 2: $\Lambda > 0$ $k = 0$ Friedmann - Lemaître model

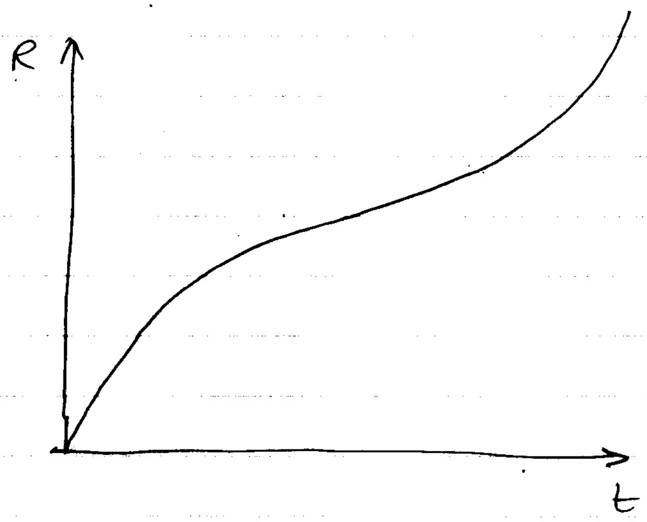
This is the currently favored model. The Friedmann equation can be solved in this case by the substitution

$$u = \frac{2\Lambda}{3C} R^3$$

and we find that

$$R^3 = \frac{3C}{2\Lambda} \left[\cosh\left((3\Lambda)^{1/2} t\right) - 1 \right]$$

Early on this model looks like Einstein de Sitter, but later begins to expand very rapidly:



Case 3: $\Lambda > 0$ $k = 0$ $c = 0$

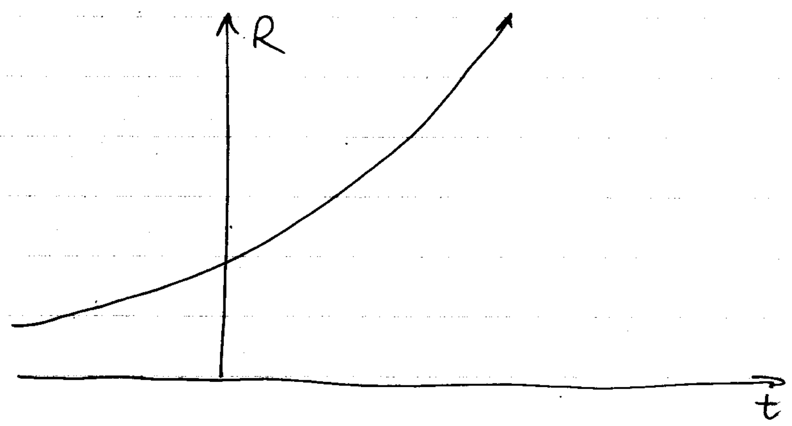
This is the de-Sitter model, it is not classified as a relativistic cosmology since it is devoid of matter. However, it is important since it describes the late behavior of Friedmann-Lemaître models & also the behavior of the universe during an ~~the~~ "inflationary epoch."

$$\frac{3\dot{R}^2}{R^2} = \Lambda$$

$$\Rightarrow \quad \frac{\dot{R}}{R} = \left(\frac{\Lambda}{3}\right)^{1/2}$$

$$\therefore \quad R = A \exp\left[\left(\frac{\Lambda}{3}\right)^{1/2} t\right]$$

where A is a constant of the integration. We can arbitrarily set $A = 1$.



The solution expands more & more rapidly with time, which appears to be the ultimate fate of our Universe.

THE END