Hence we arrive at the Schwarzschild line element

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

Some things to think about:

If $\left(1 - \frac{2GM}{c^2 r}\right) = 0$

$$\Rightarrow \quad r = \frac{2GM}{c^2}$$

and the metric appears to "break down" since $\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \longrightarrow \infty$.

Experimental Tests of G.R.

The experimental confirmation of General Relativity is extremely important. Our first solution of the field equations, the Schwarzschild metric, gives us the opportunity to look at paths in curved spacetimes via geodesics.

We will examine the advance of the perihelion of Mercury & the deflection of light by a massive object - two classical tests of the theory. First we introduce the Euler-Lagrange equations for GR to make later derivations easier.

Recall for a dynamical system described by coordinates $q^i(t)$ the Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0$$

Where $L = L(q, \dot{q})$ is the Lagrangian. The formulation for GR is precisely the same, except we use an affine parameter u rather than t:

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right) - \frac{\partial L}{\partial x^\gamma} = 0$$

I have used a Greek index to emphasize this is for GR.

We state without proof that a suitable Lagrangian for GR is given by

$$L = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \qquad \dot{} \equiv \frac{d}{du}$$

Note: For technical reasons D'Inverno refers to this as 'K'. Don't be too concerned about this, we're really using the E-L equations as a computational tool.

Differentiating the Lagrangian gives

$$\frac{\partial L}{\partial \dot{x}^\gamma} = \frac{1}{2} g_{\alpha\beta} \delta^\alpha_\gamma \dot{x}^\beta + \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \delta^\beta_\gamma = g_{\gamma\beta} \dot{x}^\beta$$

$$\frac{\partial L}{\partial x^\gamma} = \frac{1}{2} \partial_\gamma g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$$

Hence the E-L equations are

$$\frac{d}{du}\left(g_{\gamma\beta} \dot{x}^\beta \right) - \frac{1}{2} \partial_\gamma g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$$

$$\Rightarrow \quad g_{\gamma\beta} \ddot{x}^\beta + \partial_\alpha g_{\gamma\beta} \dot{x}^\alpha \dot{x}^\beta - \frac{1}{2} \partial_\gamma g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$$

By relabelling of $\dot{x}^\alpha \dot{x}^\beta$ $\quad \partial_\alpha g_{\gamma\beta} \dot{x}^\alpha \dot{x}^\beta = \frac{1}{2} \partial_\alpha g_{\gamma\beta} \dot{x}^\alpha \dot{x}^\beta + \frac{1}{2} \partial_\beta g_{\gamma\alpha} \dot{x}^\alpha \dot{x}^\beta$

Thus

$$g_{\gamma\beta} \ddot{x}^\beta + \frac{1}{2}\left(\partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} \right) \dot{x}^\alpha \dot{x}^\beta = 0$$

$$\therefore \quad \ddot{x}^\gamma + \Gamma^\gamma{}_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$$

Where $\Gamma^\gamma{}_{\alpha\beta}$ are the metric connection coefficients:

Thus we have derived the geodesic equation from this Lagrangian. Therefore we can derive a geodesic equation without actually calculating the connection coefficients explicitly. Equivalently, we now have a method to calculate them directly from the E-L equations.

We also have an analogue of a conserved momentum. If the metric does not depend on coordinates x^γ (say) then $L = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$ yields $\frac{\partial L}{\partial x^\gamma} = 0$ and thus the E-L equation ~~are~~ is

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right) = 0$$

So $\frac{\partial L}{\partial \dot{x}^\gamma}$ is a constant along the geodesic.

We can now describe timelike geodesics in the Schwarzschild geometry. Since for a timelike geodesic $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ is positive (non zero is the key point) we can choose proper time as the affine parameter. This won't be true for null geodesics for which $ds^2 = 0$.

The Lagrangian is

$$L(x^\alpha, \dot{x}^\alpha) = \frac{1}{2}\left[c^2\left(1-\frac{2m}{r}\right)\dot{t}^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)\right]$$

where $m = GM/c^2$ for ease of notation and $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \theta$, $x^3 \equiv \phi$ and $\dot{}$ denotes $\frac{d}{d\tau}$.

Since $g_{\alpha\beta}$ has no dependence on t,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^0}\right) = 0 \iff \frac{\partial L}{\partial \dot{t}} = \text{constant}$$

we find that

$$\left(1 - \frac{2m}{r}\right)\dot{t} = k \qquad \text{———} (1)$$

where k is an integration constant.

Also $g_{\alpha\beta}$ has no dependence on ϕ,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \iff \frac{\partial L}{\partial \dot{\phi}} = \text{constant}$$

If we set $\theta = \frac{\pi}{2}$, to examine motion in the equatorial plane then

$$r^2\dot{\phi} = h \qquad \text{———} (2)$$

where h is an integration constant. For r we find

$$\left(1-\frac{2m}{r}\right)^{-1}\ddot{r} + \frac{mc^2}{r^2}\dot{t}^2 - \left(1-\frac{2m}{r}\right)^{-2}\frac{m\dot{r}^2}{r^2} - r\dot{\phi}^2 = 0$$

which is rather unwieldy. We can instead rely on definition of proper time to give us something more useful.

Since $c^2 d\tau^2 \equiv g_{\alpha\beta}\, dx^\alpha dx^\beta$

We manipulate the infinitesimals to give

$$c^2 = g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

and thus

$$c^2 = c^2\left(1-\frac{2m}{r}\right)\dot{t}^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 \quad \text{—(3)}$$

This is known as a <u>metric geodesic</u>. We now have three equations for three unknowns.

If we substitute for $\dot{t}$ in (3) using (1) & (2) we get

$$c^2 = c^2\left(1-\frac{2m}{r}\right)^{-1}k^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 - \frac{h^2}{r^2} \quad \text{—(4)}$$

We now use a change of variables to connect to Newtonian theory. The Newtonian analogue of the G.R. equation we are working with is called Binet's equation and is a function of $u = 1/r$ and ϕ.

If $u = \frac{1}{r}$ then $\dot{r} = -\frac{1}{u^2}\frac{du}{d\tau} = -\frac{1}{u^2}\dot{\phi}\frac{du}{d\phi} = -h\frac{du}{d\phi}$

On multiplying through (4) by $\frac{1}{h^2}\left(1-\frac{2m}{r}\right)$ we get

$$\frac{c^2}{h^2}(1-2mu) = \frac{c^2k^2}{h^2} - \left(\frac{du}{d\phi}\right)^2 - u^2(1-2mu)$$

$$\therefore \left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2(k^2-1)}{h^2} + \frac{2mc^2u}{h^2} + 2mu^3$$

Differentiating wrt to ϕ

$$\frac{d^2u}{d\phi^2} + u = \frac{mc^2}{h^2} + 3mu^2$$

This describes the change of orbital parameter u as a function of ϕ.

Lets compare to the Newtonian version. In this case

$$\vec{F} = -m\frac{\mu}{r^2}\hat{r} \qquad (\mu \text{ to be determined})$$

Then $$m\ddot{r} = -m\frac{\mu}{r^2}\hat{r}$$

Recalling that for circular polar coordinates R, ϕ

$$\frac{d\hat{R}}{dt} = \dot{\phi}\hat{\phi} \quad \text{and} \quad \frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{R}$$

$$m\frac{d^2\vec{r}}{dt^2} = m\frac{d^2}{dt^2}(R\hat{R})$$

$$= m\left\{(\ddot{R} - R\dot{\phi}^2)\hat{R} + (R\ddot{\phi} + 2\dot{R}\dot{\phi})\hat{\phi}\right\}$$

$$= m\left\{(\ddot{R} - R\dot{\phi}^2)\hat{R} + \frac{1}{R}\frac{d}{dt}(R^2\dot{\phi})\hat{\phi}\right\}$$

Hence

$$(\ddot{R} - R\dot{\phi}^2)\hat{R} + \frac{1}{R}\frac{d}{dt}(R^2\dot{\phi})\hat{\phi} = -\frac{\mu}{R^2}\hat{R} \quad (5)$$

The scalar product with $\hat{\phi}$ gives

$$\frac{1}{R}\frac{d}{dt}\left(R^2\dot{\phi}\right) = 0$$

$$\therefore \quad R^2\dot{\phi} = \text{constant}$$

Since $\vec{L} = \vec{r} \times m\dot{\vec{r}}$

$$= R\hat{R} \times \left\{ m\dot{R}\hat{R} + mR\frac{d\hat{R}}{dt} \right\}$$

$$= mR^2\dot{\phi}\,\hat{R}\times\hat{\theta}$$

$$\Rightarrow \quad |L| = m R^2 \dot{\phi}$$

Can also show $\frac{d\vec{L}}{dt} = 0$, $\therefore \vec{L} = m\vec{h}$

where $\vec{h}$ = constant vector. Taking the modulus $|\vec{h}| = h$ we get

$$h = R^2\dot{\phi}$$ the angular momentum per unit mass

The scalar product of (5) with $\hat{R}$ gives

$$\ddot{R} - R\dot{\phi}^2 = -\frac{\mu}{R^2} \quad \text{——— (6)}$$

At this point we again substitute for $u = \frac{1}{R}$ and $\dot{R} = -h\frac{du}{d\phi}$, similarly

$$\ddot{R} = -h^2u^2\frac{d^2u}{d\phi^2},$$ Therefore (6) yields

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{h^2}$$ — the Binet's equation

Binet's equation has a solution

$$u = \frac{\mu}{h^2} + C\cos(\phi - \phi_0)$$

where C & ϕ_0 are constants. Rewriting in terms of R

$$\frac{l}{R} = 1 + e\cos(\phi - \phi_0)$$

Where $l = \frac{h^2}{\mu}$ $\quad e = \frac{Ch^2}{\mu}$

What is μ?

From the full analysis of the Newtonian two-body problem (where two objects orbit around their barycentre) one finds that $\mu = G(M_1 + M_2)$.

So comparing the relativistic & non-relativistic version we have

$$\frac{d^2u}{d\phi^2} + u^2 = \frac{mc^2}{h^2} + 3mu^2 \quad \text{—— (Rel)}$$

$$\frac{d^2u}{d\phi^2} + u^2 = \frac{\mu}{h^2} \quad \text{—— (Non-rel)}$$

Note: The relativistic version has a 'correction term', $3mu^2$. Also the mass of the orbiting particle in the GR case is negligible - it is a test particle. We must do this to avoid the particle producing local curvature within our background Schwarzschild geometry. Lastly, we solve the relativistic version via perturbation theory.