The definition of parallel propagation can be simply expressed using the absolute derivative:

$$\frac{D}{Du}\left(T^{a\cdot\cdot}_{\;\;b\cdot\cdot}\right) = 0$$

Also, for a vector field $\lambda^a(u)$ along the curves $x^a(u)$

$$\begin{aligned}\frac{D}{Du}(\lambda^a) &= \frac{dx^b}{du}\nabla_b\lambda^a \\ &= \frac{dx^b}{du}\left\{\partial_b\lambda^a + \Gamma^a{}_{cb}\lambda^c\right\} \\ &= \frac{dx^b}{du}\frac{\partial}{\partial x^b}\lambda^a + \Gamma^a{}_{cb}\lambda^c\frac{dx^b}{du} \\ &= \frac{d\lambda}{du} + \Gamma^a{}_{cb}\lambda^c\frac{dx^b}{du}\end{aligned}$$

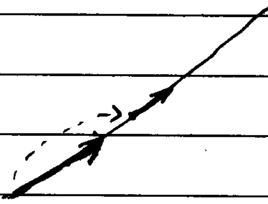
which shows the relationship of the total to the absolute derivative through the connection coefficients.

Geodesics

"Shortest distance between two points is a straight line."

We've not yet looked in detail at distance on manifolds, but we can still consider the idea of a "straight line". For a straight line the tangent vector along the curve must be transported to be a copy of itself multiplied by a function $\beta(u)$

$$\frac{D}{Du}\left(\frac{dx^a}{du}\right) = \beta(u)\frac{dx^a}{du}$$



It is straightforward to show (exercise)

$$\frac{d^2x^a}{du^2} + \Gamma^a{}_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = \beta(u)\frac{dx^a}{du}$$

this equation defines an "affine geodesic"

If we can find a parameterization, s, so as to make $\beta(u)$ vanish then

$$\frac{D}{Ds}\left(\frac{dx^a}{ds}\right) = 0$$

equivalently

$$\frac{d^2x^a}{ds^2} + \Gamma^a{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0$$

Such a parameter is called an <u>affine parameter</u>. Note: The replacement $s \to s' = as+b$ will maintain this differential equation, with $s \to s'$. The map $s \to s'$ is called an "<u>affine transformation</u>"

Geodesics play a fundamental role in GR. They define the paths that freely falling objects take as well as photons (to name but two!)

Metrics

Any non-singular rank 2 tensor field, e.g. $g_{ab}(x)$ defines a metric. In precisely the same way that $\eta_{\mu\nu}$ defined the line element in S.R., so does g_{ab} in G.R. Thus

$$ds^2 \equiv g_{ab}(x)\, dx^a dx^b$$

describes the line element on a general manifold. Note that a ~~metric~~ manifold endowed with a metric is called a Riemannian manifold.

Analogous to S.R. we define the norm of a vector by

$$X^2 = g_{ab}(x)\, X^a X^b$$

while the length of a vector is defined by $|X^2|^{1/2}$. Note that the mod is necessary since we do not require that g_{ab} be such that $X^2 > 0 \;\; \forall$ non $\underset{\sim}{0}$ $\underset{\sim}{X}$. Such a requirement would be called positive definiteness.

As in S.R. a null vector will have zero "length." This gives an unexpected result, namely that the triangle inequality

$$|\underset{\sim}{X} + \underset{\sim}{Y}| \leq |\underset{\sim}{X}| + |\underset{\sim}{Y}|$$

no longer holds.

The scaler product of two vectors is defined as $g_{ab} X^a Y^b$. This leads to the definition of the angle between two vectors:

$$\cos(X,Y) = \frac{g_{ab} X^a Y^b}{|g_{cd} X^c X^d|^{1/2} \, |g_{ef} Y^e Y^f|^{1/2}}$$

Which implies orthogonal vectors satisfy $g_{ab} X^a Y^b = 0$, and null-vectors are self orthogonal.

If we <u>define</u> g^{ab} as the inverse of g_{ab} then

$$g^{ab} g_{bc} = \delta^a_c$$

It can be shown that g^{ab} must be a tensor by considering coordinate transforms.

Together, g^{ab} and g_{ab} define index raising and lowering operations

$$g_{ab} X^b = X_a \, , \quad g^{ab} X_a = X^b$$

Intuitively if we write the norm is $X^2 = X_a X^a = g_{ab} X^a X^b$ then this result follows. The raising and lowering operations can be extended to a general tensor of type (r,s) ie $g_{ab_1} T^{b_1 b_2 \dots}{}_{c_1 \dots} = T_a{}^{b_2 \dots}{}_{c_1 \dots}$

(The mathematical operation involved involves a map of contravariant vectors to covariant forms)

These definitions should make it clear why we have emphasized the ordering of co & contravariant indices, in general $X^a{}_b \neq X^b{}_a$.

The Metric Connection

We have not yet shown how to calculate connection coefficients. There is in fact one special connection that preserves inner products under parallel transport.

Why would we want this? Consider transporting two vectors in $\mathbb{R}^n$, we would naturally expect the inner product to be the same at all places. Hence along a curve $x(u)$ $\frac{d}{du}(\underset{\sim}{X}\cdot\underset{\sim}{Y}) = 0$.

Thus we consider the inner product of two "parallel" vector fields Y^a, Z^a along a curve parameterized by u. ie.

$$\frac{D}{Du}(Y^a) = \frac{D}{Du}(Z^a) = 0$$

For their inner product

$$\frac{D}{Du}(g_{ab} Y^a Z^b) = \left(\frac{Dg_{ab}}{Du}\right) Y^a Z^b + \cancel{g_{ab}\left(\frac{DY^a}{Du}\right) Z^b} + \cancel{g_{ab} Y^a \left(\frac{DZ^b}{Du}\right)}$$

$$= \left(\frac{Dg_{ab}}{Du}\right) Y^a Z^b$$

Since by definition $\frac{D}{Du} g_{ab} = X^c \nabla_c g_{ab}$ where X is the tangent vector to the curve, we get

$$X^c(\nabla_c g_{ab}) Y^a Z^b = 0$$

This result must hold $\forall$ X^c, Y^a, Z^b and is therefore equivalent to

$$\nabla_c g_{ab} = 0$$

Expanding the covariant derivative

$$\nabla_c g_{ab} = \partial_c g_{ab} + \Gamma^d{}_{ac} g_{db} + \Gamma^d{}_{bc} g_{ad} \quad \text{— (A)}$$

We can re-label under $c \to a$, $a \to b$, $b \to c$

$$\nabla_a g_{bc} = \partial_a g_{bc} + \Gamma^d{}_{ba} g_{dc} + \Gamma^d{}_{ca} g_{bd} \quad \text{— (B)}$$

and similarly

$$\nabla_b g_{ca} = \partial_b g_{ca} + \Gamma^d{}_{cb} g_{da} + \Gamma^d{}_{ab} g_{cd} \quad \text{— (C)}$$

Calculating (A)+(B)−(C) and then using the symmetry of the connection and g_{ab}, we get

$$2\Gamma^d{}_{ca} g_{db} = \partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca}$$

We then contract with $\frac{1}{2} g^{eb}$ to get

$$\Gamma^e{}_{ca} = \frac{1}{2} g^{eb} (\partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca})$$

which directly defines the connection coefficients in terms of the metric - <u>the metric connection</u>.

<u>Fundamental Theorem of Riemannian Geometry:</u>
There exists a unique symmetric connection which preserves inner products under parallel transport (which we've just shown!)

The connection coefficients defined in this manner are also called <u>Christoffel symbols</u>.

From the earlier definition of

$$\Gamma^{e}{}_{ca} = \frac{1}{2} g^{eb}\left(\partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca}\right)$$

we can define

$$\Gamma_{abc} = \frac{1}{2}\left(\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc}\right)$$

and we can raise & lower the first index

$$\Gamma^{a}{}_{bc} = g^{ad}\Gamma_{dbc}, \qquad \Gamma_{abc} = g_{ad}\Gamma^{d}{}_{bc}$$

$\Gamma_{abc} \equiv \{bc, a\}$ is the notation used for Christoffel symbols of the 1st kind

$\Gamma^{a}{}_{bc} = \left\{ \begin{matrix} a \\ bc \end{matrix} \right\}$ is the notation for Christoffel symbols of the 2nd kind

We now need to define two very important tensors, the Riemann & Einstein tensors.
(From now on we'll use the metric connection.)

The Riemann Tensor

The structure of the covariant derivative e.g.

$$\nabla_b X_a = \partial_b X_a - \Gamma^d{}_{ab} X_d$$

suggests that it will not in general be commutative.

Let us calculate $\nabla_d \nabla_c X^a$ and $\nabla_c \nabla_d X^a$ so as to calculate the commutator $(\nabla_c \nabla_d - \nabla_d \nabla_c) X^a$

$$\nabla_d \nabla_c X^a = \partial_d(\partial_c X^a + \Gamma^a{}_{bc} X^b)$$
$$+ \Gamma^a{}_{ed}(\partial_c X^e + \Gamma^e{}_{bc} X^b)$$
$$- \Gamma^e{}_{cd}(\partial_e X^a + \Gamma^a{}_{be} X^b) \quad \text{—(A)}$$

and

$$\nabla_c \nabla_d X^a = \partial_c(\partial_d X^a + \Gamma^a{}_{bd} X^b)$$
$$+ \Gamma^a{}_{ec}(\partial_d X^a + \Gamma^a{}_{bd} X^b)$$
$$- \Gamma^e{}_{dc}(\partial_e X^a + \Gamma^a{}_{be} X^b) \quad \text{—(B)}$$

Under the assumption $\partial_c \partial_d X^a = \partial_d \partial_c X^a$ subtraction of (A) from (B) gives

$$\nabla_c \nabla_d X^a - \nabla_d \nabla_c X^a = (\partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^e{}_{bd} \Gamma^a{}_{ec}$$
$$- \Gamma^e{}_{bc} \Gamma^a{}_{ed}) X^b$$
$$+ (\Gamma^e{}_{cd} - \Gamma^e{}_{dc}) \nabla_e X^a$$

For the metric connection $\Gamma^e{}_{cd} = \Gamma^e{}_{dc}$ but it is worthwhile to note the full commutator is writte

$$\nabla_c \nabla_d X^a - \nabla_d \nabla_c X^a = R^a{}_{bcd} X^b + T^e{}_{cd} \nabla_e X^a$$

where $R^a{}_{bcd}$ defines the Riemann tensor

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^e{}_{bd} \Gamma^a{}_{ec} - \Gamma^e{}_{bc} \Gamma^a{}_{ed}$$

and $T^e{}_{cd} = \Gamma^e{}_{cd} - \Gamma^e{}_{dc}$ defines the Torsion tensor ($T^e{}_{cd} \equiv 0$ for the metric connection).

In compact notation:

$$\nabla_{[c} \nabla_{d]} X^a = \frac{1}{2} R^a{}_{bcd} X^b$$

(recall the [ ] definition includes a factor of $\frac{1}{2}$)

It is immediately clear that a necessary condition for ∇_c and ∇_d to commute is that $R^a{}_{bcd} = 0$ (this is true for tensors of all types not just X^b).

We thus define a <u>flat manifold</u> to be one for which $R^a{}_{bcd} = 0$, otherwise it is curved.

A few notes about $R^a{}_{bcd}$:

Although appearing to have N^4 components symmetries (below) reduce the number of independent components to $N^2(N^2-1)/12$

If we lower the contravariant index of $R^a{}_{bcd}$ then

$$R_{abcd} = -R_{abdc}$$
$$R_{abcd} = -R_{bacd}$$
$$R_{abcd} = R_{cdab}$$

These results can be easily checked from

$$R_{abcd} = \frac{1}{2}\left(\partial_d\partial_a g_{bc} - \partial_d\partial_b g_{ac} + \partial_c\partial_b g_{ad} - \partial_c\partial_a g_{bd}\right)$$
$$- g^{ef}\left(\Gamma_{eac}\Gamma_{fbd} - \Gamma_{ead}\Gamma_{fbd}\right)$$

The symmetric connection also leads to the cyclic identity

$$R_{abcd} + R_{adbc} + R_{acdb} = 0$$

The Riemann tensor also satisfies the Bianchi identities

$$\nabla_a R_{debc} + \nabla_c R_{deab} + \nabla_b R_{deca} = 0$$

The Riemann tensor has an important contraction: The Ricci tensor

$$R_{ab} = R^c{}_{acb}$$

It can be shown (via the cyclic identity) that the Ricci tensor is symmetric.

A further contraction gives the Ricci scalar

$$R \equiv g^{ab} R_{ab} \equiv R^a{}_a$$

The final tensor we define is the Einstein tensor G_{ab}

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}$$

From its definition it is symmetric. It can be shown from the Bianchi identities that

$$\nabla_b G_a{}^b \equiv 0$$

ie that the Einstein tensor is divergenceless.

NOTES: G_{ab} will form a critical part of the Einstein field equations. Einstein initially struggled to understand the importance of G_{ab} & spent time trying to derive field equations based on R_{ab}.

We now have the mathematical technology we need to start looking at the physical principles behind GR.

The Principles of General Relativity

There is considerable debate over the value of teaching the thought processes & principles that led Einstein to devise G.R.
We would, in principle, write down a Lagrangian & thereby derive the field equations for GR. However, such a method ignores many of the deep philosophical issues associated with the theory.

There are five distinct principles that guided Einstein in his development of GR:

(1) Mach's Principle
(2) Equivalence Principle
(3) Principle of General Covariance
(4) Principle of Minimal Gravitational Coupling
(5) Correspondence Principle

Of the list of 5, the first two have far more philosophical significance than the last three. Indeed, it has been argued that the Principle of General Covariance is "essentially empty" while others argue it can be used as the fundamental basis for deriving GR. Note that at least one of the last three principles was considered implicitly by Einstein & given a name later.
We will focus primarily on Mach's Principle & the Equivalence Principle.

Mach's Principle

In Newtonian mechanics if we accelerate a frame S' and compare to an inertial frame S, then we do not expect Newton's Laws to be obeyed. This is trivally seen for Newton's 2^{nd} Law by making the transformation

$$x \rightarrow x' = x - s, \quad y = y', \quad z = z', \quad t = t'$$

$$\therefore \quad \ddot{x}' = \ddot{x} - \ddot{s}$$

where $s = \frac{1}{2}at^2$, so that $\ddot{s} = a$ and

$$\ddot{x}' = \ddot{x} - a$$

The 2^{nd} Law in frame S is $F = m\ddot{x}$, so in frame S' we find

$$m\ddot{x}' = m\ddot{x} - ma = F - ma$$

So there is a net reduction in the force as observed by S'. This additional force, which is proportional to the mass, is called an inertial force (Coriolis force is another example). What is the physical origin of these forces? Some argue that the forces are merely the product of choosing a non-inertial reference frame and are "fictitious." Yet it is hard to argue against the physical reality of these forces! We therefore re-ask the question: What produces these forces? Equivalently, we might ask "how do we detect inertial frames?"

Newton's Bucket

Newton developed this experiment to define his concept of "absolute space." He believed that inertial forces only arise when an observer is in absolute acceleration relative to absolute space. The bucket experiment actually detects absolute rotation but is instructive nonetheless.

Suspend a bucket of water on a rope in an inertial frame. Wind-up the rope by twisting the bucket & then gently release the system to spin. Four phases of motion then arise:

(1) The bucket rotates but the water does not (surface is flat)
(2) The water eventually 'spins-up' due to friction & will form a parabola under the centripetal force
(3) Eventually the bucket stops rotating but the water will continue to rotate, and maintain an increasingly shallow parabola as the water slows down
(4) When the water returns to rest the surface is again flat

Newtonian View: the curvature of the water's surface arises because of the motion relative to absolute space.

Mach's Principle gives an alternative explanation. However, it is worth noting that the Principle is nothing more than a philosophical framework for how these forces arise. Central to the concept is the idea that motion is <u>relative</u>, ie in the absence of another body in the entire universe you cannot argue that you are in motion. Extending this idea further, Mach suggested that the origin of inertial forces was due to an <u>interaction</u> (NOT GRAVITY!!) between all the material in the Universe. Clearly, "all the material" can be viewed as being extremely close to a fixed background (mass in a shell at radius $R \propto R^2$ so mass grows as R^3). The background of matter is frequently called "fixed stars."

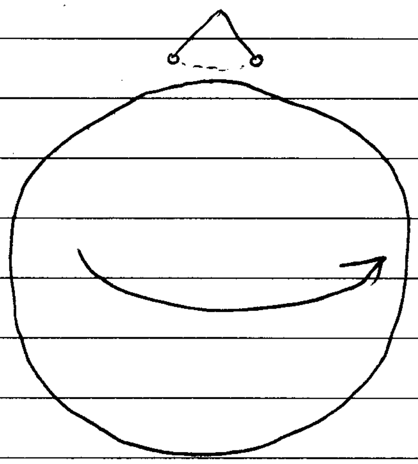
Mach's Principle thus explains the curvature of the water as being a result of rotation relative to the "fixed stars."

How is this different from Newton's viewpoint?

If the Universe were rotating relative to absolute space then under Newton's explanation the surface of a stationary bucket of water would be flat.

Under Mach's Principle, and the concept of relative ~~do~~ motion, the surface of the water would have to be curved!

What evidence do we have for this viewpoint? One simple experiment strongly supports the Machian viewpoint.



Consider a pendulum swinging above the N. Pole.

An observer on the Earth sees the pendulum rotate through 360° in one day, because the pendulum is set to swing in an inertial frame (ie not rotating relative to "absolute space").

As it turns out, the time taken for the pendulum to rotate through 360° & the time for the "fixed stars" to rotate through the same angle are equal (to experimental error).

Newtonian Theory has no requirement for this to be true, and it would merely be a coincidence (ie the fixed stars just happen to not be rotating relative to absolute space).

In Mach's Principle the times must coincide because the fixed background of stars & the pendulum are in the same state of rotation ie the fixed stars define whether a frame is rotating or not.

Does Mach's Principle make any useful statements about the nature of the interaction?

This is difficult to answer. We might reasonably expect that since inertial forces are proportional to an objects mass, that the total inertial effect depends upon the total mass of the fixed stars. In this case, the mass of the Earth & Sun will contribute a negligible amount to inertial properties & therefore these forces should be the same in all places (at least locally!) What about all directions? This requires isotropy which as we'll see is a requirement of some metrics in cosmology. It has also been confirmed that inertial forces are the same in arbitrary directions to 1 part in 10^{18}

Currently there is debate over the true value of Mach's Principle. It is embodied in "Brans-Dicke" theory, however this theory has to be so-similar to GR that it is an unnecessary complication. However, it cannot be denied that Einstein used Mach's Principle as a guiding principle in his derivation of GR.

Principle of Equivalence

This was probably the key idea for Einstein in developing GR. He called it "the happiest thought of my life". The realization being that an observer falling from a house - at least in his immediate vicinity - fells no gravitational field. If the observer were to drop some "bodies" during the fall then (air resistance aside) those objects would remain stationary relative to the falling observer, regardless of the composition of the objects.

Before we look at the Principle of Equivalence in more detail, it is useful to consider mass in Newtonian theory. There are actually three distinct masses

Inertial { (1) Inertial mass, m^I (ie as in $\vec{F} = m^I \vec{a}$)

Gravitational { (2) Passive gravitational mass, m^P (as in $\vec{F} = -m^P \nabla\phi$)

(3) Active gravitational mass, m^A (as in $\phi = -\frac{G m^A}{r}$)

For two distinct objects (ie hammer & feather!)

$$m_1^I \vec{a}_1 = \vec{F}_1 = -m_1^P \nabla\phi$$

$$m_2^I \vec{a}_2 = \vec{F}_2 = -m_2^P \nabla\phi$$

Thanks to Galileo & the Apollo astronauts(!) we know: $\vec{a}_1 = \vec{a}_2$ experimentally, so we get

$$\frac{m_1^I}{m_1^P} = \frac{m_2^I}{m_2^P}$$

We can take this ratio to be unity by choosing appropriate units (without loss of generality). Then

inertial mass = passive gravitational mass

For the active gravitational mass, given two objects the potentials will be

$$\phi_1 = -\frac{G m_1^A}{r_1} \qquad \phi_2 = -\frac{G m_2^A}{r_2}$$

at a distance r. If the objects are allowed to interact, then the forces as measured by the passive gravitational masses will be

$$\vec{F} = m_1^P \nabla \phi_2 \qquad \text{(at point 1)}$$
$$\vec{F}_2 = m_2^P \nabla \phi_1 \qquad \text{(at point 2)}$$

By Newton's 3rd Law, and using $\nabla \equiv \hat{r}\frac{\partial}{\partial r}$ we find

$$\frac{m_1^P}{m_1^A} = \frac{m_2^P}{m_2^A}.$$

Thus we again imply an equality

passive gravitational mass
= active gravitational mass.

Thus under Newtonian theory

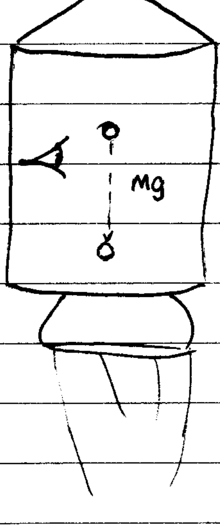
$$m = m^P = m^A = m^I$$

Lift Experiments

Let us conside 4 different thought experiments (gedänken in German). Einstein collectively called them "The lift experiments."
In the lift we place an observer, along with apparatus for measuring acceleration of objects in the lift.
The question we ask is: "Can the observer determine the state of motion of the lift based on these experiments?"

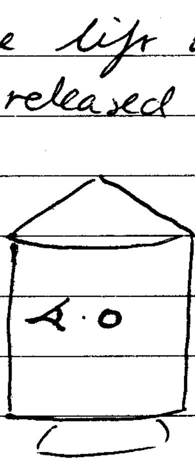
Case 1: "Rocket lift"

We put a giant rocket on the back of the lift & take it to a point far away from other gravitating bodies. The rocket is then uniformly accelerated, with acc^n g. The observer sees objects fall to the bottom of the lift with acc^n g.

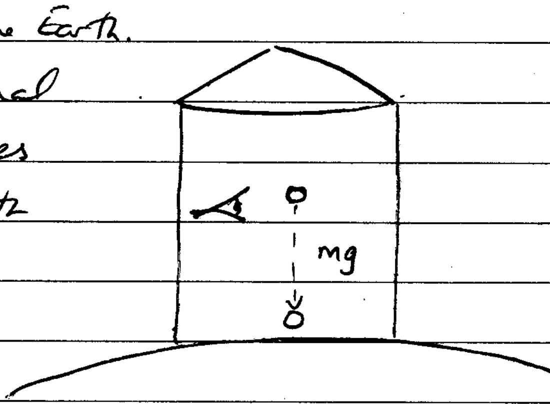


Case 2: "Uniform lift"

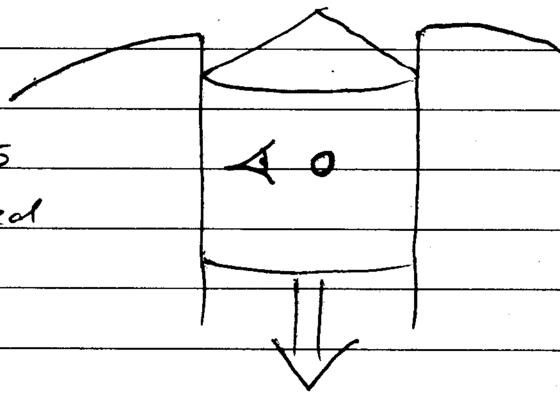
The rocket motor is switched off. The lift is then in a state of uniform motion. A released object stays at rest relative to the rest of the lift.



Case 3: Lift on surface of the Earth.
Note we ignore orbital & rotational motions in this case. Observer sees objects fall to the ground with acc^n g.



Case 4: Free-falling lift
We place the lift in an evacuated shaft & allow it to fall under gravity. A released body again stays at rest relative to the observer.



Consider cases 1 & 3. To the observer they are completely indistinguishable. Similarly cases 2 & 4 appear exactly the same to the observer!

This lead Einstein to the idea that inertial mass and gravitational mass must be equivalent. It implies we can create a gravitational field by acceleration (Think of case 1). Of course though the acceleration is confined to the rocket & the field is local. Nonetheless, to the observer it is a gravitational field.

We can show how these ideas relate to curved space times in the following manner. The equation of a free particle in Minkowski coords is

$$\frac{d^2 x^\alpha}{d\tau^2} = 0$$

If we chose a more general coordinate system we must calculate

$$\nabla_X\left(\frac{dx^\alpha}{d\tau}\right) = X^\delta \nabla_\delta\left(\frac{dx^\alpha}{d\tau}\right) = 0$$

$$= X^\delta \left\{ \partial_\delta \frac{dx^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\delta} \frac{dx^\beta}{d\tau} \right\}$$

$$= \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\delta} \frac{dx^\beta}{d\tau} \frac{dx^\delta}{d\tau}$$

$\Gamma^\alpha{}_{\beta\delta}$ will be the metric connection associated with the $g_{\alpha\beta}$ of the more general coordinate system. These extra terms are therefore associated with <u>inertial forces</u>.

Since the Principle of Equivalence requires both inertial & gravitational forces to be equivalent, both forces must be describable using $\Gamma^\alpha{}_{\beta\delta}$. Relating back to the equation of the metric connection. $\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} + \partial_d g_{bc})$,

This implies we should think of the metric $g_{\alpha\beta}$ as being a potential-like term.

Principle of General Covariance

The Principle of S.R states that all inertial observers are equivalent. Einstein was convinced that any observer should be capable of discovering the laws of physics. This should seem a logical necessity to us sitting on the Earth in a non-inertial frame!

Principle of General Relativity: All observers are equivalent.

Note, this does not imply that observers will uncover similar physical laws as preferred coordinate frames may exist in which "fictitious forces" vanish. Therefore it is necessary to formulate theories so as to be invariant under coordinate transformations:

Principle of General Covariance: The equations of physics should have tensorial form.

This directly explains why we took such pains to learn tensorial algebra & calculus. Again, it has been argued this statement is comparatively empty (any theory can be written in a tensorial form) but it was very important to Einstein.

Principle of Minimal Gravitational Coupling

If we were to generalize equations from SR to GR, the most simple generalization under the Principle of General Covariance would be to replace partial derivatives with covariant derivatives:

$$\text{e.g.} \quad \partial_b T^{ab} = 0 \rightarrow \nabla_b T^{ab} = 0$$

However, we would equally well add in other tensorial terms that are a function of the curvature tensor, e.g.

$$\partial_b T^{ab} \longrightarrow \nabla_b T^{ab} + g^{be} R^a{}_{bcd} \nabla_e T^{cd}$$

Since $R^a{}_{bcd} = 0$ in the Minkowski coordinates of GR, this would be a legitimate generalization. We therefore adopt a simplicity principle (C.F. Occam's Razor)

Principle of Minimal Gravitational Coupling:
No terms explicitly containing the curvature tensor should be added in making the transition from the special to the general theory.

Einstein relied on this principle implicitly & it was only stated later.