PHYS4390 - Assignment 5

Due in Fri, Mar 30, 2012 (Time allowed=two weeks)

Show all working.

(1) Consider the Minkowski metric,

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

written in spherical polar coordinates (t, r, θ, ϕ),

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(a) Calculate the non-zero connection coefficients for this metric.
(b) Introduce the *retarded time* coordinate $u = t - r$ and show that,

$$ds^2 = du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Write down the g^{ab} and g_{ab} in matrix form using (u, r, θ, ϕ) coordinates. Verify that the hypersurfaces $u = constant$ are null hypersurfaces and find a vector field l^a that is orthogonal to these hypersurfaces.
(c) Repeat (b) using the *advanced time* coordinate $v = t + r$.
(d) Derive the line element for (u, v, θ, ϕ) and write down g_{ab} and g^{ab} in matrix form.

(2) In the notes we showed that coordinate time can be confusing when attempting to describe the infall of a particle into a Schwarzschild black hole. Using coordinate time, we stated that that close to the event horizon the path follows

$$r - 2m = (r_0 - 2m)e^{-(t-t_0)/2m}.$$

Prove this result using the geodesic for an infalling particle (HINT: consider a small parameter $\epsilon = 1 - r/2m$).

(3) Describing orbits around a black hole. This is a very important topic because we can now make observations of stars orbiting around Sgr A*. In this question we'll develop some of the techniques used.
(a) We showed early on that for the 4-velocity, u^α, (assuming c=1),

$$u_\alpha u^\alpha = 1.$$

Write this equation out using the Schwarschild metric, but in the equatorial plane.
(b) The zeroth component of the 4-velocity describes energy. If we define svector ξ^α and η^α such that $\xi^\alpha = (1, 0, 0, 0)$ and $\eta^\alpha = (0, 0, 0, -1)$, then calculate

$$e = \xi_\alpha u^\alpha \qquad l = \eta_\alpha u^\alpha$$

for the Schwarzschild spacetime .
(c) Hence show that $u_\alpha u^\alpha$ is equivalent to,

$$\varepsilon = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r),$$

where,

$$\varepsilon = (e^2 - 1)/2, \qquad V_{eff}(r) = -\frac{m}{r} + \frac{l^2}{2r^2} - \frac{ml^2}{r^3} = -\frac{m}{r} + \frac{l^2}{2r^2}\left(1 - \frac{2m}{r}\right).$$

(d) This result is essentially equivalent to the result for Newtonian theory. Beginning from,

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{GMm}{r} \text{ and } l = mr^2\dot{\phi}$$

show that in Newtonian theory,

$$\frac{1}{2}m\dot{r}^2 = E - U_{eff}(r)$$

and give the equation for U_{eff}.
(e) Returning to the result for the relativistic theory, let's use it to determine what the longest possible proper time an observer can take between passing the event horizon and hitting the singularity is. Setting $l = 0$ produces the longest time - why?
(f) Entering the event horizon with the lowest velocity possible will maximize the proper time, hence evaluate it.

(4) The perfect-fluid energy momentum tensor is

$$T^{\alpha\beta} = (\rho_0 + p)u^\alpha u^\beta - pg^{\alpha\beta}$$

where ρ_0 is the proper density and p is the scalar pressure field. By considering the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0,$$

in Minkowski space, show that for a pressureless fluid the time and space components of this equation reduce to the three dimensional continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{v}),$$

and the (3d, pressureless) Navier-Stokes equation,

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}\right) = 0.$$

(HINT: think carefully about going from the 4-velocity to the three velocity and how this impacts the density.)

(5) This is a question about how fields (specifically scalar fields without mass) evolve in an expanding spacetime. Note, this was set as an exam question in 2007.
(a) Show that the contraction of the connection coefficient $\Gamma^\alpha_{\alpha\beta}$ (*i.e.* summation implied) is

$$\Gamma^\alpha_{\alpha\beta} = \frac{1}{2}g^{\alpha\delta}\partial_\beta g_{\alpha\delta}$$

(b) Prove that if the metric is diagonal, this expression can be written

$$\Gamma^\alpha_{\alpha\beta} = \frac{1}{\sqrt{g}}\partial_\beta\sqrt{g}$$

where g is the determinant of the metric tensor. (HINT: the determinant has a simple form if the metric tensor is diagonal, and you'll need to use an identity for the derivative of a log.)
(c) Rewrite

$$\nabla_\alpha\nabla^\alpha\Phi = 0,$$

using the FLRW metric for a spatially flat universe,

$$ds^2 = c^2dt^2 - R^2(t)(dx^2 + dy^2 + dz^2),$$

and the earlier results to derive an equation for $\ddot{\Phi}$ in terms of $\dot{\Phi}$, R, and $\dot{R}$. (HINT: You'll first need to expand one of the covariant derivatives in the wave equation and substitute using the result from part(b). The square root of the metric determinant will be complex, but the complex factors should cancel.)
(d) Solve the equation for $\dot{\Phi}$ as a function of $R(t)$ for the case where Φ is spatially homogeneous (HINT: *i.e.* $\nabla^2\Phi = 0$).