PHYS4390 - Assignment 1

Due in Class on Wed, Feb 1, 2012 (Time allowed=two weeks)

Show all working.

(1) Cosmic rays hitting the top of the atmosphere (say 10 km up) generate showers of very short-lived particles. Typically the proper decay rate of these particles is less than 1 microsecond (10^{-6} s), yet many of them reach ground level. Estimate the difference between their speed and the speed of light.

(2) S and S' are two inertial frames connected by a boost along the x-axis. At time t=0 photons are seen by an observer in S to pass through the origin with velocity $\mathbf{c} = (c_x, c_y, 0)$. Show from first principles (*not* simply using formulae for the transformation of velocities) that in frame S' the photons have velocity $\mathbf{c}' = (c'_x, c'_y, c'_z)$ where

$$c'_x = (c_x - v)L_1,$$
$$c'_y = c_y L_2,$$
$$c'_z = 0$$

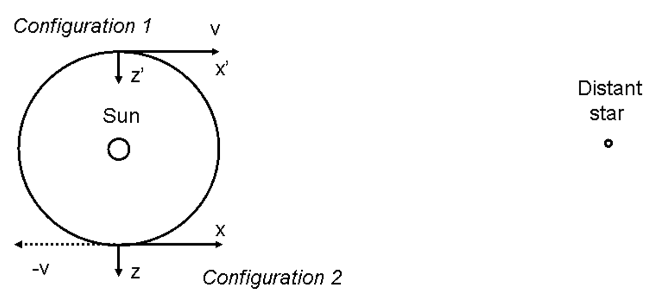
where for a Galilean boost $L_1 = L_2 = 1$, while for a Lorentz boost,

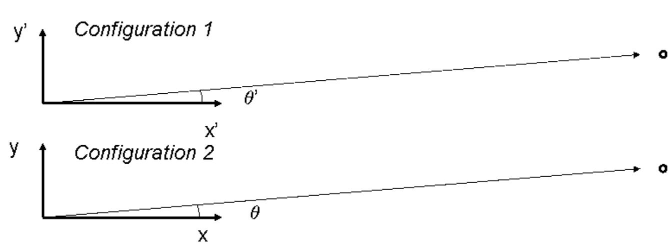
$$L_1 = \left(1 - \frac{c_x v}{c^2}\right)^{-1},$$

$$L_2 = \frac{L_1}{\gamma}$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Check that in the case of a Lorentz boost $|\mathbf{c}| = |\mathbf{c}'|$.
Next consider the Earth orbiting the Sun. Relative to a distant star (sufficiently distant that we can ignore the size of the Earth's orbit), at one point of the orbit the velocity will be $\mathbf{v}_{Earth}$ towards the star, while a half period later the velocity will be -$\mathbf{v}_{Earth}$ (away) from the star. The star also makes an angle θ relative to the xy-plane of the system:





This corresponds to the same physical situation as described in the first part of the question with $v \simeq 2v_{Earth}$. Firstly, resolve the velocity vector **c** associated with the light from the star that reaches Earth into its components which can be labelled c_x, c_y, c_z (which will be functions of θ) in the frame S. Given these components, derive expressions for $\tan\theta'$ in the frame S' as a function of $\sin\theta$, $\cos\theta$, v and c, for both the Galilean and Lorentz transformations. Give a general formula for $\tan\theta'$ (similar to that used in the earlier part of the question where L_1 could take different values) where a factor A incorporates whether a Galilean or Lorentz transformation is used.

(3) Show that if $(x_A^0, x_A^1, x_A^2, x_A^3)$ and $(x_B^0, x_B^1, x_B^2, x_B^3)$ are any two points in 4D spacetime, then

$$\eta_{\nu\mu}(x_B^\mu - x_A^\mu)(x_B^\nu - x_A^\nu)$$

which can be written in matrix-vector form (T=transpose)

$$(\mathbf{x}_B - \mathbf{x}_A)^T \eta (\mathbf{x}_B - \mathbf{x}_A),$$

is invariant under the *inhomogeneous* Lorentz Transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + b^\mu, \quad b^\mu \text{ are constant, equivalently } \mathbf{x}' = \Lambda\mathbf{x} + \mathbf{b},$$

where the matrix for $\Lambda^\mu{}_\nu$ is given by $\Lambda = R_2\Lambda_B R_1$ where R_1, R_2 are spatial rotations, and Λ_B is a Lorentz boost in the x direction (we mentioned this point briefly in the lectures). (HINT: Use the matrix notation and think about how the 4x4 matrices corresponding to the spatial rotations will be constructed, for rotation matrices what does $R^T R$ equal?)
Deduce that the speed of light is invariant under this transformation.

(4) The equation of a straight line in a 4D spacetime can be written in the form

$$\frac{d^2 x^\mu}{ds^2} = 0 \tag{1}$$

where $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.
(a) What is the equation of this line in terms of the parameter s?
(b) Show that for this equation in 4D space, the particle moves with uniform three-velocity in 3D space.
(c) If the line determined by equation 1 passes through the point with coordinates x_i^μ when $s = s_i$ and i=1 or 2, show that

$$s_{21} = s_2 - s_1 = (\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu)^{1/2},$$

where $\Delta x^\mu = x_2^\mu - x_1^\mu$.

(5) Consider the following coordinate transformation from cartesian coordinates (x,y) to a new set of coordinates (μ, ν):

$$x = \mu\nu \qquad y = \frac{1}{2}(\mu^2 - \nu^2)$$

(a) Sketch curves of constant μ and constant ν in the x,y plane.
(b) Transform the line element $ds^2 = dx^2 + dy^2$ into (μ, ν) coordinates.
(c) Do the curves of constant μ and constant ν intersect at right angles?
(d) Find the equation of a circle of radius R centered at the origin in terms of μ and ν.
(e) Calculate the ratio of the circumference to the diameter of the circle using (μ, ν) coordinates. Do you get the correct answer?