PHYS 3300 - Classical Mechanics

Website: www.ap.smu.ca/~thacker/teaching/3300/3300.html
Prerequisites: Introduction to Modern Physics
Analytical Mechanics

Builds skills for: Fluid Dynamics
General Relativity
Quantum Mechanics II

New Curriculum for 2008:

(1) Calculus of Variations (~3 lectures)
(2) Lagrangian Dynamics (~4 lectures)
(3) Rigid Body Dynamics (~4 lectures)
(4) Special Relativity (~2 lectures)
(5) Hamiltonian Dynamics (~4 lectures)
(6) Chaos (~4 lectures)
(7) Perturbation Theory (~2 lectures)

Book: Goldstein "Classical Mechanics" 3rd edition
(co-authors Poole & Safko).

- Terrific resource that can be used in grad school. Do not be intimidated by the content. We will choose sections & chapters that are at the appropriate level.

Marking scheme: 20% assignments (best 4 of 5)
30% mid term
50% Final

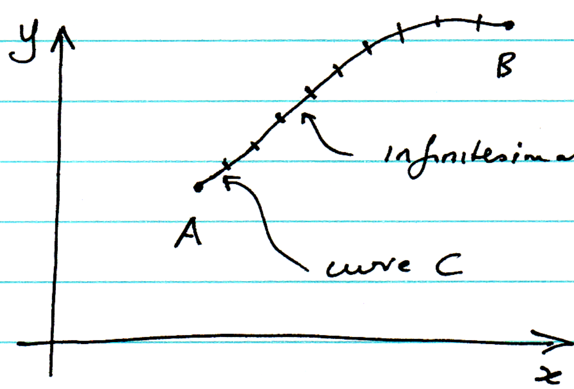
Office hours: 9-10 Thurs & Fri, other times by appointment

Today: Review of line integrals & introduction to calculus of variations

Line integral recap:
Consider a piece of string that lies in the x-y plane and makes a curve, C. The mass per unit length of the string is $f(x,y)$, so that the total mass of the string is the sum of all the infinitesimal lengths, ds, each multiplied by the local mass per unit length:

$$\text{Mass} = \int_C f(x,y)\, ds$$

We can draw this situation for an arbitrary curve C as follows:



To calculate the integral we need to
(1) Parameterize the curve C with a variable
(2) Evaluate the infinitesimal path length as a function of the variable
(3) Integrate to get the total mass

We will assume that the curve is both continuous (since it is one piece of string) and smooth, so that the first derivative exists.

Let us assume that the curve can be parameterized by a variable t (we might be able to choose x, but for now we want to be general). Since C defines a curve in the x-y plane we can think of it as being defined by a position vector $\vec{r} = (x, y)$. Since it is parameterized by t, we have

$$\vec{r} \equiv \vec{r}(t) = (x(t), y(t))$$

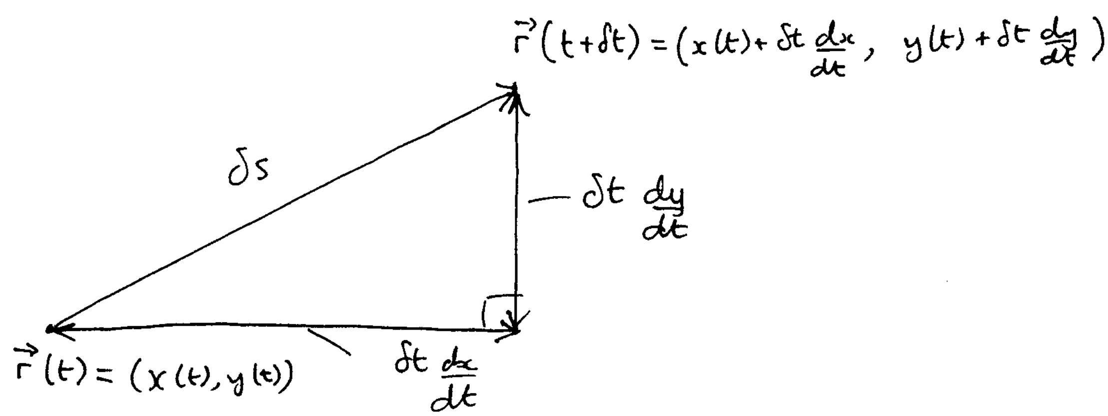
A point a distance δt from $\vec{r}(t)$ will then be

$$\vec{r}(t+\delta t) = (x(t+\delta t), y(t+\delta t))$$

Applying Taylor expansions to the x & y coordinates

$$\vec{r}(t+\delta t) = \left(x(t) + \delta t \frac{dx(t)}{dt}, \; y(t) + \delta t \frac{dy(t)}{dt} \right)$$

Thus we can represent the distance between $\vec{r}(t)$ and $\vec{r}(t+\delta t)$ as follows:



In the infinitesimal limit $\delta t \to dt$, the distance ds is given by

$$ds^2 = dt^2 \left(\frac{dy}{dt}\right)^2 + dt^2 \left(\frac{dx}{dt}\right)^2$$

and hence

$$ds = \sqrt{y'^2 + x'^2}\, dt$$

where $'$ denotes $\frac{d}{dt}$.

Thus the mass per infinitesimal piece can now be written down:

$$dm = f(x(t), y(t)) \sqrt{y'^2 + x'^2}\, dt$$

and the total mass is

$$M = \int_c f(x,y)\, ds = \int_{t_1}^{t_2} f(x(t), y(t)) \sqrt{y'^2 + x'^2}\, dt$$

Example: (Easy). Suppose the string is laid out in a semi-circle of radius 1 and the density function is $f(x,y) = y$

[Question: If the string was laid out around a circle centered on the origin why would the density not be physically sensible?]

Since the curve is a semi-circle, our choice for the parameter t will be the polar angle θ. θ will then range from 0 to π. The coordinates of the curve are given by

$$x(\theta) = \cos\theta \qquad \text{and} \qquad \frac{dx}{d\theta} = -\sin\theta$$
$$y(\theta) = \sin\theta \qquad \qquad \frac{dy}{d\theta} = \cos\theta$$

hence

$$M = \int_0^{\pi} \sin\theta \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta$$

$$= \int_0^{\pi} \sin\theta \, d\theta = 1.$$

The same approach can be used in 3d, this time the z-coordinate would also be parameterized by t and the line element must be adjusted accordingly.

Functionals, Path Integrals & the Calculus of Variations

We spend a great deal of time in introductory physics studying how one physical quantity is proportional to another. These "scaling relationships" are powerful for determining overall behaviour, but they rarely describe the complete process.
In practice, many processes depend upon much more than a single number, for example the distance travelled in a time t is dependent upon the path taken, one can only write $s = vt$ for straight line motion at uniform velocity. If we consider an object moving under gravity on a surface, the distance travelled in time T will be a function of the overall shape of the surface.

A functional takes an input function, say $y(x)$, defined over a range of x, (the domain) and outputs a number that depends upon the $y(x)$ for all x's in the domain.

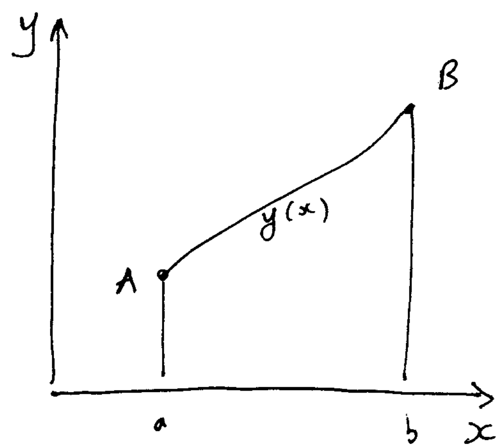
For our ~~pp~~previous example, the time taken, T, is a functional of the shape of the surface (and gravity).

There are many old examples of problems that associate a single number to a curve from a family of possibilities:

Queen Dido of Carthage: She was promised as much land as could be fitted within the boundary of a bull's hide. She cut the hide into strips & we hope that she arranged the strips into a circle!

Newton: Calculated what shape minimizes air resistance.

As a first example of the calculus of variations let us consider Archimedes famous problem: what curve $y(x)$ gives the shortest distance between two points? (Yes, we all know the answer but the problem is quite illustrative).



The distance along the path is

$$l = \int_A^B ds$$

If we parameterize the curve using $y(x)$, then we can write

$$ds^2 = dy^2 + dx^2$$

$$\Rightarrow \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Hence

$$l = \int_a^b \sqrt{1 + y'^2}\, dx \equiv \int_a^b F(y')\, dx$$

where $'$ now denotes $\frac{d}{dx}$.

To solve Archimedes' problem we need to minimize l for some function $y(x)$. Our constraints are that the curve must start at A $(x=a)$ and finish at B $(x=b)$ - this describes the domain of x.

When finding the min/max of a function $f(x)$ we look for $\frac{df}{dx} = 0$. If we consider what happens to $f(x)$ as $\cancel{x} \rightarrow x + \delta x$ then Taylor expansion gives

$$f(x + \delta x) = f(x) + \left.\frac{df}{dx}\right|_x \delta x + \frac{(\delta x)^2}{2!} \left.\frac{d^2 f}{dx^2}\right|_x + \ldots$$

If we assume δx to be small, then neglecting higher order terms we can write

$$f(x + \delta x) = f(x) + \delta f(x) = f(x) + \left.\frac{df}{dx}\right|_x \delta x$$

So the change in $f(x)$ given by $\delta f(x)$ is just

$$\delta f(x) = \left.\frac{df}{dx}\right|_x \delta x$$

Thus at a max/min, which we will call an extremum from now on, then $\delta f(x) = 0$.

We can apply this idea to functionals as well.

The extremum of l will be given by a <u>function</u> y that under $y \to y + \delta y$ produces a change in l of $\delta l = 0$.

Since $l \equiv \int_a^b F(y')\,dx$ we can write

$$\delta l = l[y' + \delta y'] - l[y'] = \int_a^b F(y' + \delta y')\,dx - \int_a^b F(y')\,dx$$

Expand $F(y' + \delta y') = F(y') + \delta y' \dfrac{\partial F}{\partial y'}$

$$\Rightarrow \qquad \delta l = \int_a^b \frac{\partial F}{\partial y'}\,\delta y'\,dx$$

Given that $y' = \dfrac{dy}{dx} \Rightarrow \dfrac{d}{dx}(y + \delta y) = \dfrac{dy}{dx} + \dfrac{d}{dx}(\delta y)$

hence $\delta y' = \dfrac{d}{dx}(\delta y)$

$$\Rightarrow \qquad \delta l = \int_a^b \frac{\partial F}{\partial y'}\,\frac{d}{dx}(\delta y)\,dx$$

Apply integration by parts

$$\delta l = 0 = -\int_a^b \delta y \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx + \left[\frac{\partial F}{\partial y'}\,\delta y\right]_a^b$$

Since b & a are both fixed in x, but also both fixed in the y coordinate too (ie A & B cannot move up/down or left/right) the variation in the curve, δy, must be zero at A & B.

Thus

$$\left[\frac{\partial F}{\partial y'}\,\delta y\right]_a^b = 0 \qquad \text{and}$$

$$0 = -\int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y \, dx \qquad \text{——— (A)}$$

Now, since (A) must be true for all possible changes in y, ie all δy, then the only way the integral can be zero is if

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

If we integrate once,

$$\frac{\partial F}{\partial y'} = \text{constant} = c$$

If we substitute for $F(y') = (1+y'^2)^{1/2}$ Then

$$\frac{\partial F}{\partial y'} = \frac{y'}{(1+y'^2)^{1/2}} = c$$

Quick rearrangement yields

$$y' = \frac{c}{\sqrt{1-c^2}} = d_\ell$$

which is another constant. Hence integrating y'

$$y = d_\ell x + f$$ where f is a constant of integration.

This is the equation of a straight line – as expected!

Calculus of Variations

We previously considered the straightforward problem of determining the shortest path between two points. The integral we considered was of the form

$$I[y] = \int_a^b F(y'(x))\, dx \qquad y' = \frac{dy}{dx}$$

and there was no explicit dependence on $y(x)$, only the derivative $y'(x)$ appears. Let us now consider a more general case where

$$I[y] = \int_a^b F(y(x), y'(x))\, dx$$

Our previous derivation will be made more rigorous. Suppose that we define the change in the path, δy, in terms of a parameter α and another function $\eta(x)$. In this case the paths under consideration can be written

$$\text{"}y(x) + \delta y(x)\text{"} \equiv y(x) + \alpha\eta(x) \equiv y(x,0) + \alpha\eta(x) \equiv y(x,\alpha)$$

As $\alpha \to 0$ so we recover $y(x)$. The function $\eta(x)$ must be constrained to be zero at the end points since these are fixed, hence $\eta(a) = \eta(b) = 0$.

We also require that $y(x)$ & $\eta(x)$ be
(1) continuous & non-singular
(2) have continuous first & second derivatives

Since α is a single parameter, not a function, the integral is now a <u>function</u> of α:

$$I(\alpha) = \int_a^b F(y(x,\alpha), y'(x,\alpha))\, dx \qquad \text{——— (A)}$$

While it remains a functional of $y(x)$ & $\eta(x)$ we specifically want to consider the variation of $I(\alpha)$ as $\alpha \to 0$, as this is equivalent to $\delta y \to 0$.

Since α is small, we can again consider the variation in $I(\alpha)$ using a Taylor series:

$$I(\alpha) \simeq I(0) + \alpha \left.\frac{dI}{d\alpha}\right|_{\alpha=0}$$

So finding the extremum will be equivalent to solving

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0$$

Thus we differentiate both sides of (A):

$$\frac{dI}{d\alpha} = \int_a^b \left\{ \frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha} \right\} dx \qquad (x \text{ is not a function of } \alpha)$$

Since $y(x,\alpha) = y(x,0) + \alpha\,\eta(x)$

$$\Rightarrow \quad \frac{\partial y}{\partial \alpha} = \eta(x) \quad \& \quad \frac{\partial y'}{\partial \alpha} = \eta'(x)$$

Substituting gives

$$\frac{dI}{d\alpha} = \int_a^b \left\{ \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right\} dx$$

Since $\eta' = \frac{d\eta(x)}{dx}$ we can apply integration by parts to the second term

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial F}{\partial y}\eta \, dx + \left[\frac{\partial F}{\partial y'}\eta\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\, dx$$

but $\left[\frac{\partial F}{\partial y'}\eta\right]_a^b = 0$ since $\eta(a) = \eta(b) = 0$ because end points of the curve are fixed.

$$\therefore \quad \frac{dI}{d\alpha} = \int_a^b \left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}\eta(x)\, dx$$

As stated earlier, the extremum is given by

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0, \quad \text{hence}$$

$$\int_a^b \left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right\}\eta(x)\, dx = 0$$

implies we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

as $\eta(x)$ is an arbitrary function.

This is the Euler-Lagrange equation & is extremely important in mechanics. While it appears simple, the precise form of $F(y, y')$ will determine how complex the resulting dynamics are. In general the equation leads to a second order non-linear differential equation. Generally this equation will be hard to solve directly, but some simplifications exist.

There are two special cases that help simplify the E-L equations.

(1) $F \equiv F(x, y')$ so that F does not depend on y

In this case $\frac{\partial F}{\partial y} = 0$ and hence

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad \frac{\partial F}{\partial y'} = \text{constant}$$

This is what we observed for the shortest path problem. We now have a first order DE to solve which should obviously be much easier to handle.

(2) Suppose $F \equiv F(y, y')$ so that it has no explicit dependence on x, ie $\frac{\partial F}{\partial x} = 0$.

Starting from the E-L equation we have

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$\Rightarrow \quad y'\frac{\partial F}{\partial y} = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \qquad \text{——— (A)}$$

Since $\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$

$$\Rightarrow \quad y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) - y''\frac{\partial F}{\partial y'}$$

Substitute on LHS of (A) to get

$$y'\frac{\partial F}{\partial y} = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) - y''\frac{\partial F}{\partial y'} \qquad \text{——— (B)}$$

The next step is to write out $\frac{dF}{dx}$:

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{dy}{dx}\frac{\partial F}{\partial y} + \frac{d^2y}{dx^2}\frac{\partial F}{\partial y'}$$

$$= \frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'}$$

Since we have specified $\frac{\partial F}{\partial x} = 0$, we have

$$\frac{dF}{dx} = y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'}$$

Substituting in (B) yields

$$\frac{dF}{dx} = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)$$

$$\Rightarrow \quad \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$$

Integrate once to give

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}$$

So again we have derived a first order DE rather than second.

Examples:

Conside the shortest path problem of the 1^{st} lecture. We had to find the extremum of

$$I = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \Rightarrow \quad F = (1+y'^2)^{1/2}$$

Thus $\frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial y'} = \frac{y'}{(1+y'^2)^{1/2}}$

hence the E-L equation gives

$$\frac{d}{dx}\left(\frac{y'}{(1+y'^2)^{1/2}}\right) = 0$$

$$\Rightarrow \quad \frac{y'}{(1+y'^2)^{1/2}} = \text{constant} \quad \text{(as before)}.$$

Note that this result is particularly straightforward because F has no dependence on $y(x)$.

For the second example we shall conside the "Brachistochrone*" problem first posed (and solved) by Johann Bernoulli in 1696. The problem can be stated as:

Determine the curve through points A & B that will allow a ball rolling on the curve under gravity to complete the journey from A to B in the shortest possible time.

[* Brachistos = the shortest, chronos = time]

Potential energy of ball due to gravity is $-mgy$
Kinetic energy of ball is $\frac{1}{2}mv^2$

Since energy is conserved we must have

$$\frac{1}{2}mv^2 - mgy = \text{constant}$$

Scale y so that the constant is zero. (we need only add a constant in this case).

To work out the time taken we have to sum all the times to traverse each infinitesimal path length ds at speed v:

$$\text{Since} \quad \frac{ds}{dt} = v \quad \Rightarrow \quad dt = \frac{ds}{v}$$

$$\text{hence} \quad T = \int_0^T dt = \int_A^B \frac{ds}{v} \quad \text{between } A \,\&\, B$$

ds will again be given by decomposing into x & y components:

$$ds^2 = dx^2 + dy^2$$
$$\Rightarrow \quad ds = \sqrt{1 + y'^2}\, dx$$

For the velocity we substitute for v using

$$\frac{1}{2}v^2 = gy$$

$$\Rightarrow \quad v = \sqrt{2gy}$$

Thus $T = \int_A^B \frac{ds}{v} = \int_{x_A}^{x_B} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx = \int_0^{x_B} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx$

The function $F(y, y')$ is given by $F(y,y') = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$

Since there is no explicit dependence on x, we can use our earlier result that

$$F - y'\frac{\partial F}{\partial y'} = \text{constant} = c \text{ ——— (d)}$$

Calculating $\frac{\partial F}{\partial y'}$ we get

$$\frac{\partial F}{\partial y'} = y'(1+y'^2)^{-1/2}(2gy)^{-1/2}$$

and subsituting into (d) gives

$$\frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - y'\frac{y'}{\sqrt{2gy}\sqrt{1+y'^2}} = c$$

$$\Rightarrow \qquad \frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} = c$$

On squaring both sides & rearranging we get

$$y'^2 = \frac{k}{y} - 1 \Rightarrow y' = \sqrt{\frac{k}{y} - 1} \quad \text{where } k = \frac{1}{2c^2 g}$$

We can solve this equation by trying a parametric substitution:

Let $\quad y = k\sin^2\theta = \frac{k}{2}(1 - \cos 2\theta)$

$$\Rightarrow \quad \frac{dy}{d\theta} = k\sin 2\theta$$

Then since $\frac{dy}{dx} = \frac{d\theta}{dx}\frac{dy}{d\theta} = \left(\frac{dx}{d\theta}\right)^{-1}\frac{dy}{d\theta}$

$$\Rightarrow \quad \sqrt{\frac{k}{k\sin^2\theta} - 1} = \left(\frac{dx}{d\theta}\right)^{-1} . \; k\sin 2\theta$$

$$\frac{1}{\tan\theta} = \left(\frac{dx}{d\theta}\right)^{-1} . \; k\sin 2\theta$$

$$\Rightarrow \quad \frac{dx}{d\theta} = k\sin 2\theta \;\tan\theta$$

$$= 2k\sin\theta\,\cancel{\cos\theta}\,\frac{\sin\theta}{\cancel{\cos\theta}}$$

$$= 2k\sin^2\theta = k(1-\cos 2\theta)$$

Integrate to get

$$x = \int^{\theta} k(1-\cos 2\theta)\, d\theta$$

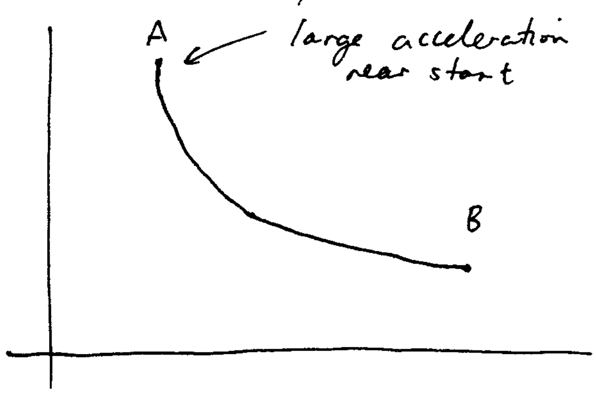
$$= k\left(\theta - \frac{1}{2}\sin 2\theta\right)$$

(constant set to zero by starting at $x=0$)

So we have a parametric solution:

$$\left.\begin{aligned} y &= \frac{k}{2}(1-\cos 2\theta) \\ x &= k\left(\theta - \frac{1}{2}\sin 2\theta\right) \end{aligned}\right\} \quad \text{where } \theta \in \left[0, \frac{\pi}{2}\right]$$

These are the parametric equations of an (inverted) cycloid.



A cycloid corresponds to the locus of a point on the circumference of a circle rotating along a fixed line.