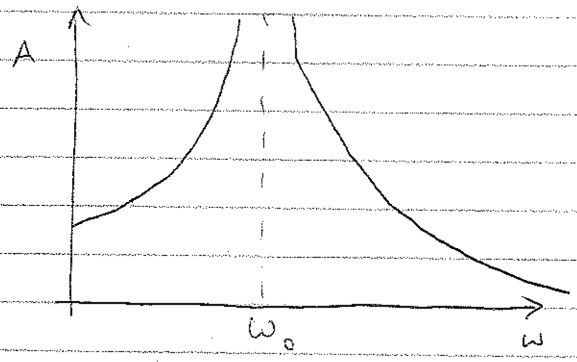
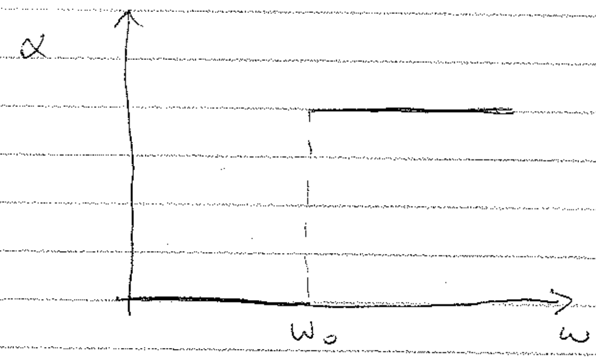
Forced Oscillations cont.

Recall we considered undamped forced oscillators and found for the amplitude & phase:





Now let's consider adding a damping force.

Recall damped free oscillator equation:

$$m\ddot{x} = -kx - b\dot{x}$$

With forcing we have

$$m\ddot{x} + kx + b\dot{x} = F_0 \cos \omega t$$

$$\Rightarrow \quad \ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} = \frac{F_0}{m}\cos\omega t$$

and using $\gamma = \frac{b}{m}$ $\omega_0 = \sqrt{\frac{k}{m}}$

$$\Rightarrow \quad \ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

Lets solve via complex exponential methodology:

$\Rightarrow$ Solve $\ddot{z} + \gamma\dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{j\omega t}$

We'll use a trial solution $z = Ae^{j(\omega t - \delta)}$

The $-\delta$ is OK, we could have put $+\alpha$ instead, it's just a helpful definition here.

Substitute using:

$$\ddot{z} = -\omega^2 A e^{j(\omega t-\delta)}$$
$$\dot{z} = j\omega A e^{j(\omega t-\delta)}$$
$$z = A e^{j(\omega t-\delta)}$$

$$\Rightarrow \quad -\omega^2 A \cancel{e^{j(\omega t-\delta)}} + \gamma j\omega A \cancel{e^{j(\omega t-\delta)}} + \omega_0^2 A \cancel{e^{j(\omega t-\delta)}} = \frac{F_0}{m} \cancel{e^{j\omega t}}$$

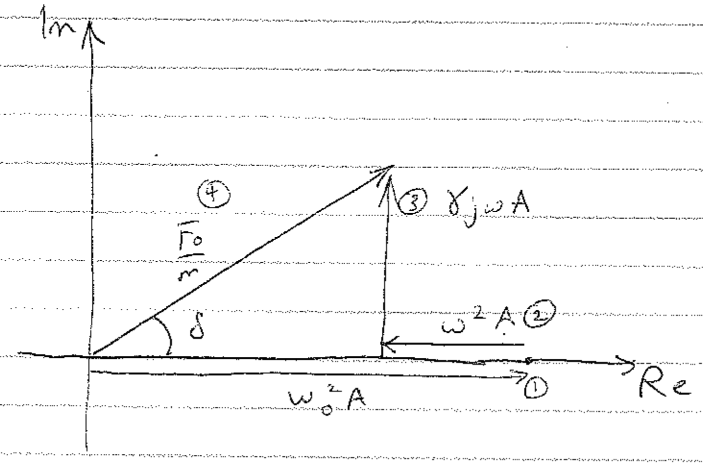
$$\Rightarrow \quad -\omega^2 A e^{-j\delta} + \gamma j\omega A e^{-j\delta} + \omega_0^2 A e^{-j\delta} = \frac{F_0}{m}$$

multiply through by $e^{j\delta}$

$$-\omega^2 A + \gamma j\omega A + \omega_0^2 A = \frac{F_0}{m} e^{j\delta}$$

$$\Rightarrow \quad \underbrace{(\omega_0^2 - \omega^2) A + \gamma j\omega A}_{\text{complex}} = \underbrace{\frac{F_0}{m} e^{j\delta}}_{\text{complex}} \qquad (*)$$

Let's consider what this looks like in the complex plane:



If we expand the RHS of (*) then

$$(\omega_0^2 - \omega^2) A + \gamma j\omega A = \frac{F_0}{m} \{\cos\delta + j\sin\delta\}$$

and now we can equate the real & imaginary parts.

$$(\omega_0^2 - \omega^2) A = \frac{F_0}{m} \cos\delta \qquad \text{(Real part)}$$

$$j \gamma \omega A = j \frac{F_0}{m} \sin\delta \qquad \text{(Im part)}$$

$$\Rightarrow \quad \gamma \omega A = \frac{F_0}{m} \sin\delta$$

Q: Can we form an equation for δ that doesn't include the amplitude A?

Yes! Divide the imaginary part by the real part:

$$\Rightarrow \quad \frac{\gamma\omega}{\omega_0^2 - \omega^2} = \frac{\sin\delta}{\cos\delta} = \tan\delta$$

Notice F_0 & m are taken out too!

Thus the phase is set solely by γ, ω, ω_0 ie b, k, m, since $\gamma = \frac{b}{m}$, $\omega_0 = \sqrt{\frac{k}{m}}$

Next we need to look at the amplitude A:

Q: Can we write down formula for A that does not depend on δ?

Yes! Two ways to think about it. From real & imaginary parts we could write:

$$\cos\delta = \frac{Am}{F_0}(\omega_0^2 - \omega^2)$$

$$\sin\delta = \frac{\gamma\omega A m}{F_0}$$

and then use $\cos^2\delta + \sin^2\delta = 1$

OR Can just think of applying Pythagoras in the complex plane

Hypotenuse $= F_0/m$ opposite $= \gamma\omega A$

adjacent $= (\omega_0^2 - \omega^2) A$

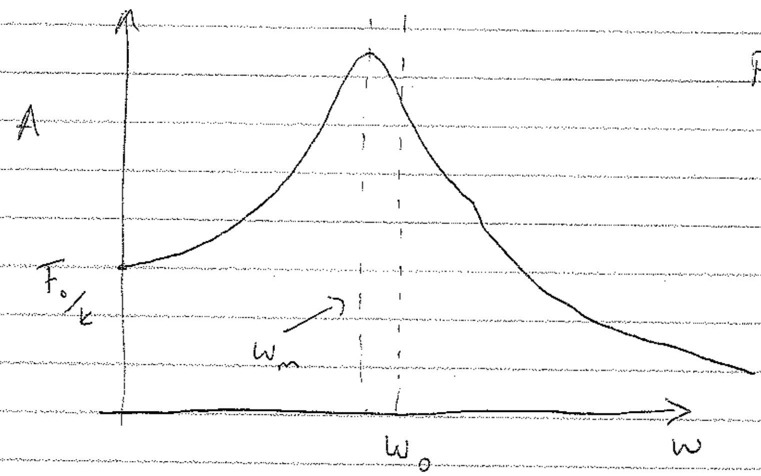
both approaches give

$$\left(\frac{F_0}{m}\right)^2 = (\omega_0^2-\omega^2)^2A^2 + (\gamma\omega A)^2$$

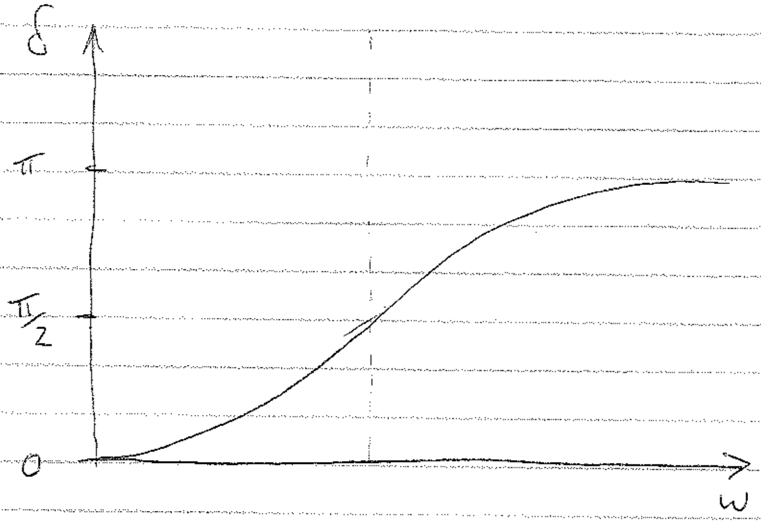
$$= A^2\left[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2\right]$$

and rearranging:

$$A = \frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2\right]^{1/2}}$$



Peak is actually slightly before ω_0 and called "ω_m"



$$\frac{\gamma\omega}{\omega_0^2-\omega^2} = \tan\delta$$

$\delta = \frac{\pi}{2}$ when $\omega_0^2-\omega^2=0$

$\tan\delta = 0$ at $\omega=0$

$\tan\delta\to 0$ as $\omega\to\infty$

So in damped systems phase changes smoothly, as does the amplitude.

How does changing the resistive term b, or γ, change these results?

$\Rightarrow$ Physically: What does more or less damping do?

Recall we used $Q = \frac{\omega_0}{\gamma}$ as a measure of damping. Large $Q \longrightarrow$ little damping
Small $Q \longrightarrow$ lots of damping

Thus we'll rewrite $A = \frac{F_0}{m} \frac{1}{\left[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right]^{1/2}}$

using Q by subbing $\gamma = \frac{\omega_0}{Q}$

$$\Rightarrow \quad A = \frac{F_0}{m} \frac{1}{\left[(\omega_0^2-\omega^2)^2 + \left(\frac{\omega_0^2\omega^2}{Q^2}\right)\right]^{1/2}}$$

$$= \frac{F_0}{m} \frac{1 \cdot \quad 1/\omega_0\omega}{\left[(\omega_0^2-\omega^2)^2 + \left(\frac{\omega_0^2\omega^2}{Q^2}\right)\right]^{1/2} \frac{1}{\omega_0\omega}}$$

$$= \frac{F_0}{m} \frac{1/\omega_0\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

$$= \frac{F_0}{m} \frac{1}{\omega_0^2} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

since $\omega_0^2 = \frac{k}{m}$

$$= \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

So now we have an expression in terms of Q & ω, ω_0 ($+ F_0/k$)

What about δ?

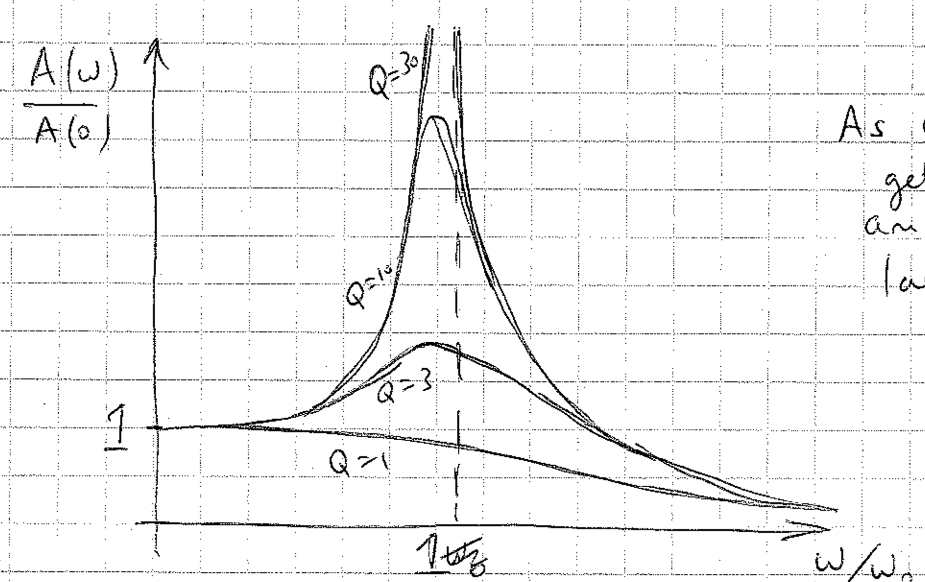
Since $$\tan\delta = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

$$= \frac{\omega_0\omega/Q}{\omega_0^2 - \omega^2} \quad \text{using } Q = \frac{\omega_0}{\gamma}$$

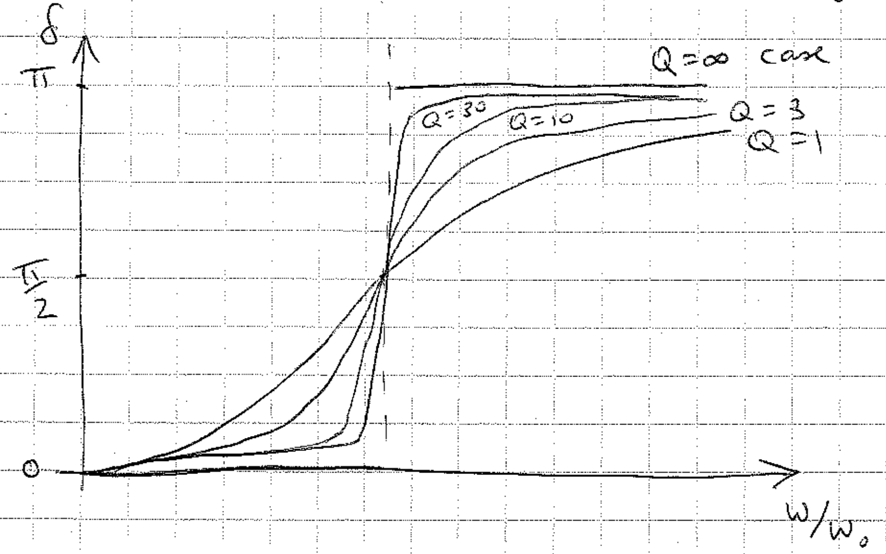
$$= \frac{1/Q}{\left(\frac{\omega_0}{\omega}\right)^2 - \left(\frac{\omega}{\omega_0}\right)^2}$$

What happens as we vary Q?

Lets draw the y-axis as $\frac{A(\omega)}{A(0)}$ & x as ω/ω_0



As Q increases & $\frac{1}{Q^2}$ gets smaller so the amplitude peak gets larger



Key observations:

(1) Even though system oscillates at angular frequency ω, ω_0 still plays a role in setting the <u>amplitude</u>

(2) The overall "sharpness" of the amplitude peak is set by $Q = \cancel{\tfrac{1}{2}}\frac{\omega_0}{\gamma} = \frac{\omega_0 m}{b}$