Forced Vibrations & Resonance

Recap: Thus far we've considered simple HM where

$$m\ddot{x} = -kx \quad \text{(for the spring case)} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$

in the case of damping we added a velocity dependent term:

$$m\ddot{x} = -kx - bv$$

Saw that this modified both frequency & amplitude,

$$x = Ae^{-\gamma t/2}\cos(\omega t + \alpha)$$

where $\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$ $\qquad \gamma = \frac{b}{m}$

We haven't considered the case where there is an external driving force,

$$F = F_0 \cos \omega t \qquad \text{(periodic driving force)}$$

So that

$$m\ddot{x} = -kx - bv + F_0 \cos(\omega t)$$

$$\Rightarrow \quad m\ddot{x} + kx + b\dot{x} = F_0 \cos(\omega t)$$

Key questions:

(1) At what frequency will the system oscillate?
(2) Associated period?
(3) What will the amplitude be?

To begin consider the case with no damping

$$m\ddot{x} + kx = F_0 \cos(\omega t)$$

Guess at a solution of form $x = C\cos\omega t$

Substitute in, then get

$$-m\omega^2 C \cancel{\cos(\omega t)} + kC \cancel{\cos(\omega t)} = F_0 \cancel{\cos(\omega t)}$$

$$\therefore \quad -m\omega^2 C + kC = F_0$$

$$\Rightarrow \quad C = \frac{F_0}{k - m\omega^2}$$

$$\therefore \quad C = \frac{F_0/m}{k/m - \omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2}$$

where $\omega_0^2 = k/m$ as defined earlier. So this trial solution works.
$\Rightarrow$ oscillation at $\underline{\omega}$.

Interpretation:

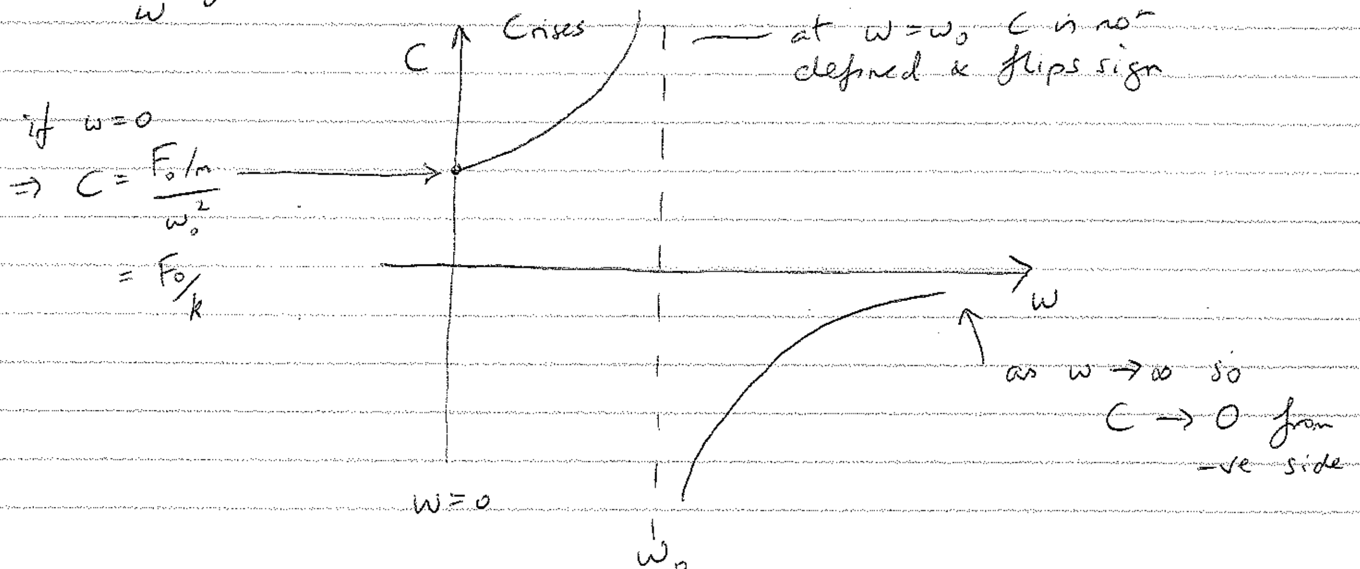
(1) ω_0 & ω do not have to be the same

$\Rightarrow$ if the system is driven it does not have to vibrate at its natural frequency.

Q: What happens to the amplitude when the frequency of the driving force & the natural frequency coincide?

A: As $\omega \to \omega_0$ then C grows in amplitude at the exact matching point it is not defined

Very instructive to consider how C varies with ω



We've thus shown that

(1) Drive system with a sinusoidal driving force will oscillate at the frequency of the driver (ω)

(2) Their amplitude will depend upon the difference between the natural frequency & the driving frequency.

The sign flipping of the amplitude feels somewhat strange.

Suppose we consider $x = A\cos(\omega t + \alpha)$ but let $\alpha = 0$ if $\omega < \omega_0$
$\alpha = \pi$ if $\omega > \omega_0$

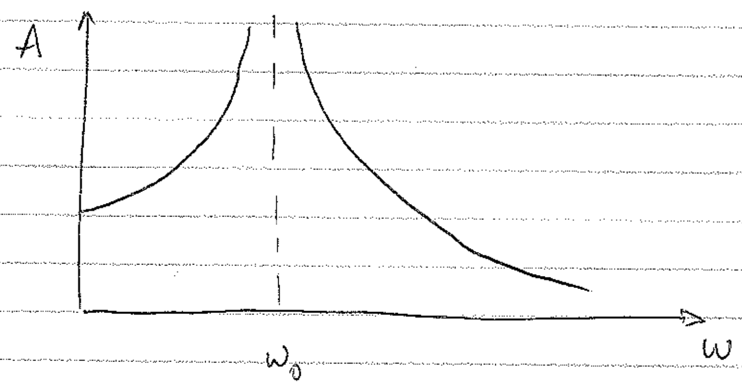
This gives us

$$x = \begin{cases} |C| \cos \omega t & \text{if } C > 0 \text{ ie } \omega < \omega_0 \\ |C| \cos(\omega t + \pi) & \text{if } C < 0 \text{ ie } \omega > \omega_0 \end{cases}$$

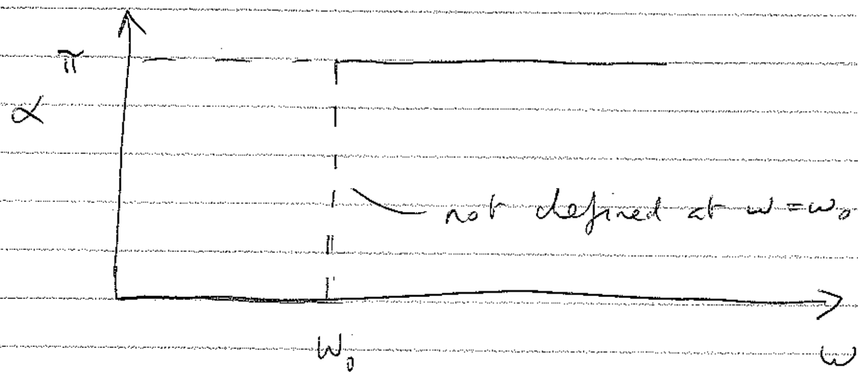
So we quickly see we're equating $|C| = A$ & it is always +ve.

The sign flipping is now associated with a phase shift of π.

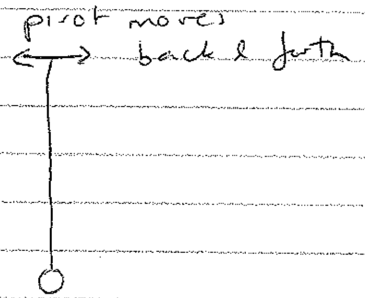
So we can now re-draw the "A" vs ω diagram:



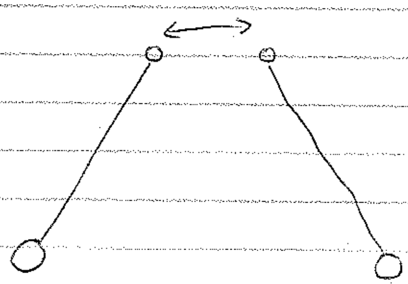
but now we also need to include a phase shift diagram:

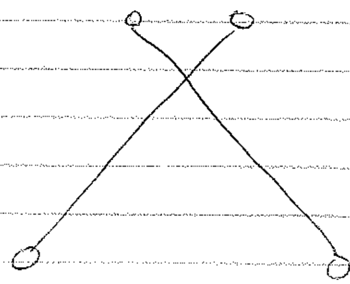


This change in phase can be most easily seen in a driven pendulum



For $\omega < \omega_0$ | For $\omega > \omega_0$





We can also apply the complex exponential method for this undamped driven case.

Recall we were working with the eqn of motion:

$$m\ddot{x} + kx = F_0 \cos \omega t$$

In the complex exponential form we need to solve

$$m\ddot{z} + kz = F_0 e^{j\omega t}$$

and then projecting the solution on to the real axis will give us what we want.

$\therefore$ Try $z = Ae^{j(\omega t + \alpha)}$ where again $x = Re\{z\}$

Substituting we find

$$m(-\omega^2) Ae^{j(\omega t+\alpha)} + kAe^{j(\omega t+\alpha)} = F_0 e^{j\omega t}$$

$$\Rightarrow \quad -m\omega^2 Ae^{j\alpha} + kAe^{j\alpha} = F_0$$

divide through by m & $e^{j\alpha}$

$$\Rightarrow \quad \left(-\omega^2 + \frac{k}{m}\right) A = \frac{F_0}{m} e^{-j\alpha}$$

recall we defined $\omega_0^2 = \frac{k}{m}$

$$\Rightarrow \quad (\omega_0^2 - \omega^2) A = \frac{F_0}{m}(\cos -\alpha + j \sin -\alpha)$$

$$= \frac{F_0}{m}(\cos\alpha - j\sin\alpha)$$

Need to have both the real & imaginary parts solved.

$$\Rightarrow \quad 0 = -\frac{F_0}{m}\sin\alpha$$ for imaginary part

true for $\alpha = 0, n\pi$

For real part:

$$(\omega_0^2 - \omega^2) A = \frac{F_0}{m} \cos\alpha$$

but since we looked at α being set by $\alpha = 0, \pi, 2\pi$ etc we take $\alpha = 0, \pi$

$\therefore \cos\alpha = \pm 1$

$$\Rightarrow \quad A = \frac{\pm F_0/m}{(\omega_0^2 - \omega^2)}$$ for the two values of α

While $\omega < \omega_0$ A is +ve
As $\omega > \omega_0$ numerator sign flips, but so does denominator so A is +ve again

Thus

$$x = Re\{z\} = Re\{A e^{j(\omega t + \alpha)}\}$$

$$= A \cos(\omega t + \alpha)$$

$$= \begin{cases} A \cos \omega t & \omega < \omega_0 \\ A \cos(\omega t + \pi) & \omega > \omega_0 \end{cases}$$