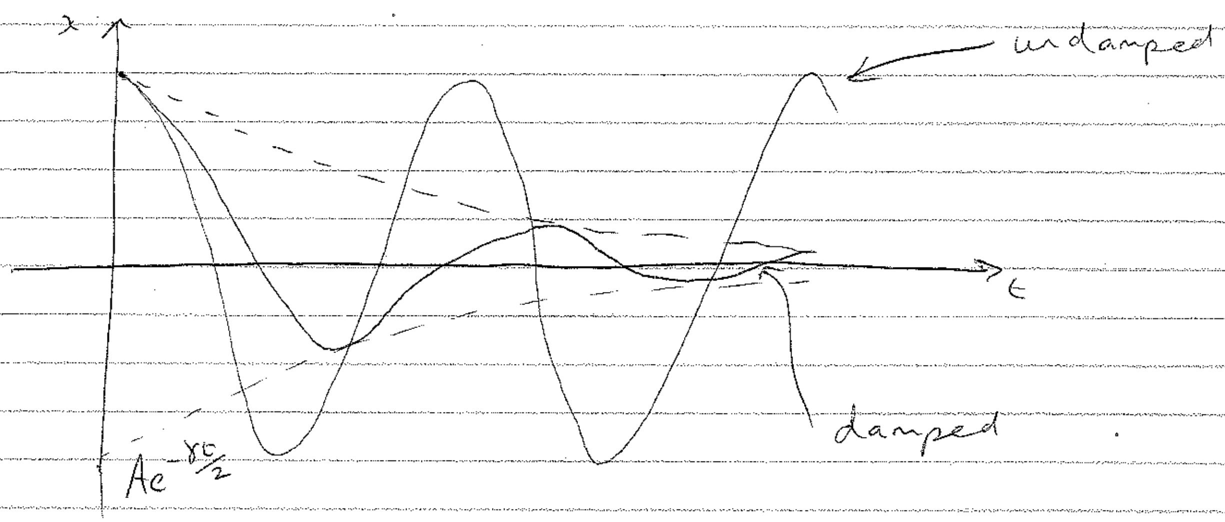
Damped Harmonic Oscillators cont

Recap: Established that damped vibration decays exponentially

Damped vibrations oscillate at a lower frequency



Q: How would we calculate total energy of this system?

$$E_{TOT} = E_{PE} + E_{KE}$$

We would do this using $x = Re\, Ae^{j(pt + \alpha)}$

But lets do a simpler case. At each peak of the extension

$$E_{KE} = 0$$

So $E_{TOT} = E_{PE}$

So given amplitude we can estimate the total energy provided that the amplitude is not changing much over one cycle.

Q: When is the most energy lost in a cycle?

Consider: What is the energy loss coming from?
Ans: Resistive force

What is this at peak extension when $v = 0$?
$bv = 0$ so it is minimized

bv is maximized when the velocity is maximized within the cycle, so around the middle of oscillation.

This means energy loss will not simply be an exponential, it will be modified by an oscillatory term (although the total E can <u>never grow</u>).

Coming back to peak to peak change:

$$E_{TOT} = \frac{1}{2} k x_{max}^2 = \frac{1}{2} k \{ A e^{j(pt_{max} + \alpha)} \}^2$$

$$= \frac{1}{2} k \{ A e^{-\gamma t/2} \cos(\omega t_{max} + \alpha) \}^2$$

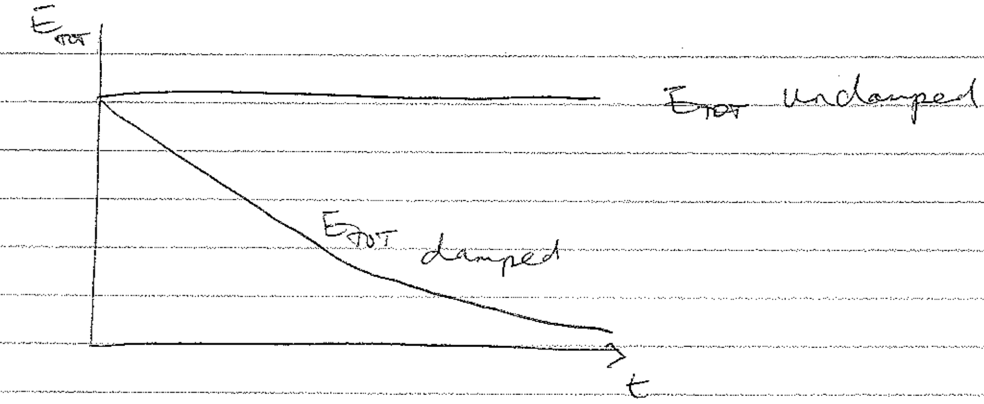
$$= \frac{1}{2} k A^2 e^{-\gamma t} \underbrace{\cos^2(\omega t_{max} + \alpha)}_{= 1 \text{ at max extension}}$$

$$= \frac{1}{2} k A^2 e^{-\gamma t} \equiv \frac{1}{2} k A_0^2 e^{-\gamma t}$$

We often write $A = A_0$ to remind us that A_0 is the initial amplitude at the starting time.

Thus $E_{TOT} \simeq E_{TOT}(0) e^{-\gamma t}$

where $E_{TOT}(0) = \frac{1}{2} k A_0^2$



We've shown that γ & ω_0 govern the evolution of the system.

But these are set by $\gamma = \frac{b}{m}$, $\omega_0 = \left(\frac{k}{m}\right)^{1/2}$

$\Rightarrow$ b, k, m set all the parameters of the solution (modulo starting conditions)

γ is a measure of the rate of energy loss
ω_0 is a " " the undamped oscillation rate

Consider $Q = \frac{\omega_0}{\gamma}$ Q for "quality"

Small γ $\Rightarrow$ little dissipation $\Rightarrow$ bigger Q
Large γ $\Rightarrow$ lots of " $\Rightarrow$ smaller Q

We can rewrite $\omega = \omega_0^2 - \frac{\gamma^2}{4} = \omega_0^2 \left\{ 1 - \frac{1}{4Q^2} \right\}$

For large Q: $\omega \sim \omega_0^2 \{ 1 - \text{v. small} \} \sim \omega_0^2$

as expected, and we can write

$$x \simeq A_0 e^{\frac{-\omega_0 t}{2Q}} \cos(\omega_0 t + \alpha)$$

See p. 68 on how Q relates to the number of cycles of oscillation compared to the amount of amplitude decay.

Thus far we considered light damping where $\gamma\omega_0^2 > \frac{\gamma^2}{4}$

What if $\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} < 0$ ie $\frac{\gamma}{2} > \omega_0 = \frac{k}{m} \Rightarrow \gamma > 2\omega_0$

Intuition: This is implying that the resistive force term is having more impact on motion than the restoring force.

$\Rightarrow$ Motion doesn't oscillate.

So for this case write the eqn for the real part of p as

$$n^2 = \omega_0^2 - \frac{\gamma^2}{4} = -\underbrace{\left\{\frac{\gamma^2}{4} - \omega_0^2\right\}}_{\text{is now +ve since we are considering } \frac{\gamma}{2} > \omega_0}$$

$$\therefore \quad n = \pm j\left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2} = j\beta$$

where $\beta = \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}$

Since the solution we had was
$$x = Re\, z$$
$$= Re\, A e^{j(pt+\alpha)}$$
$$= Re\, A e^{-st} e^{j(nt+\alpha)}$$

Subbing for n gives

$$x = Re\, A e^{-st} e^{\pm\beta t} e^{j\alpha}$$

$$\therefore \quad x = Re\, A e^{-\left[\frac{\gamma}{2} \mp \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}\right]t}$$

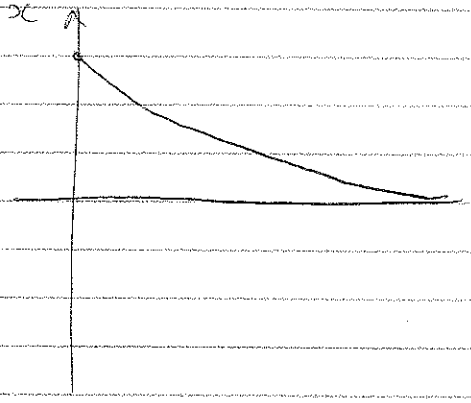
ignoring constant $e^{j\alpha}$ part

Most general solution includes both possible β!

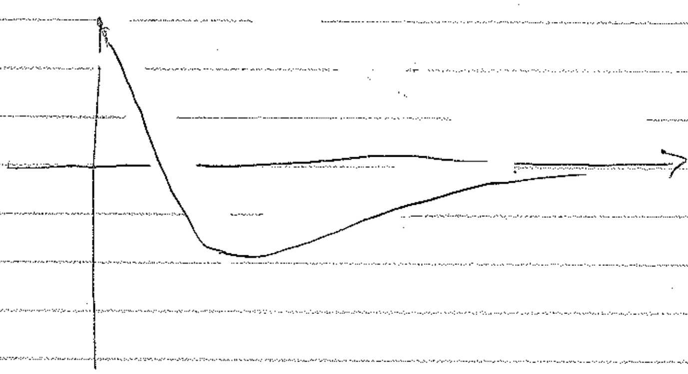
$$x = A_1 e^{-\left(\frac{\gamma}{2}+\beta\right)t} + A_2 e^{-\left(\gamma/2 - \beta\right)t}$$

No oscillatory part!

The different parts of the solution allow us to adjust for different starting velocities:



low initial velocity



high initial velocity

$$x = e^{-\gamma t/2}\left\{A_1 e^{-\beta t} + A_2 e^{\beta t}\right\}$$

Last case: What if $\omega_0^2 = \frac{\gamma^2}{4}$

The two parts of $x = A_1 e^{-(\frac{\gamma}{2}+\beta)t} + A_2 e^{-(\frac{\gamma}{2}-\beta)t}$ become equivalent.

Still need two adjustable constants to account for starting position & velocity changes.

You can verify $x = (A + Bt)e^{-\gamma t/2}$

solves the original equation of motion if $\gamma = 2\omega_0$

This is known as "critical damping"

No overshoot! Produces most rapid decay to zero. Very useful in engineering (think of door closer)

Hence we classify the behaviour as follows:

$$\omega_0^2 > \gamma^2/4 \Rightarrow \text{"under damped"}$$

$$\omega_0^2 < \gamma^2/4 \Rightarrow \text{"overdamped"}$$

$$\omega_0^2 = \gamma^2/4 \Rightarrow \text{"critically damped"}$$