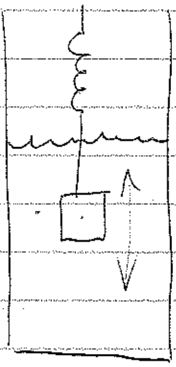
Decay of free vibrations

- Vibrations do not continue forever
- The loss of energy is related to the concept of damping

Where does the energy go?



Consider spring-mass system in water

- Where does the energy go?

Into water motion

- If you can hear it, it's losing energy!

Resistive forces are often $\propto v, v^2$ so that

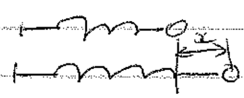
$$\vec{F}_d = -b_1 v\hat{v} - b_2 v^2 \hat{v}$$

$^* \; v \ll \frac{b_1}{b_2}$

In many cases the dependence on the v^2 term is small at low velocities*, so we often write

$$\vec{F}_d \simeq -b_1 v\hat{v} = -b_1 \vec{v} = b\vec{v}$$ (dropping subscript)

This means the equation of motion is now:



$$\underbrace{F = m\ddot{x} = -kx}_{\text{Same as before}} - bv$$

damping force resists motion

Thus $m\ddot{x} = -kx - b\dot{x}$

$$\Rightarrow \quad \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

let $\gamma = \frac{b}{m}$ $\omega_0^2 = \frac{k}{m}$ then

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$

So ω_0^2 is analogous to $\omega = \frac{k}{m}$ for the undamped case.

Units: Interesting to think about the units of γ & ω_0:

$$\omega_0^2 \equiv \frac{k}{m} = \frac{F/x}{m} \Rightarrow \text{Units are } \frac{MLT^{-2}/L}{M} = T^{-2}$$

So ω_0 has units of frequency

For γ: $$\gamma \equiv \frac{b}{m} = \frac{F/v}{m} \Rightarrow \text{units are } \frac{\cancel{M}LT^{-2}/LT^{-1}}{\cancel{M}} = T^{-1}$$

So γ <u>also</u> has units of frequency

⇒ It tells us something about time dependence in the solution.

Lets solve the equation of motion. Use the complex exponential approach where we take the real part of z as x. Firstly, consider

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = 0 \qquad \text{where } z = Ae^{j(pt+\alpha)}$$

Then $\dot{z} = jpAe^{j(pt+\alpha)} = jpz$

$$\ddot{z} = j^2 p^2 z = -p^2 z$$

Substituting back into equation

$$\{-p^2 + jp\gamma + \omega_0^2\} Ae^{j(pt+\alpha)} = 0$$

To be true at <u>all times</u> bracket must be zero ie

$$-p^2 + jp\gamma + \omega_0^2 = 0$$

But this is a complex equation with real & imaginary parts.
Note we can't set $p=0$ as then there is no time evolution.

If $\gamma = 0$ then we don't have any damping, so it must be non-zero, + it is real.

Together, these imply p must have real & imaginary parts (ω_0 is real).

Thus set $p = n + js$. Then for the imaginary part:

$$p^2 = n^2 + 2jns - s^2$$

So $-p^2 + jp\gamma + \omega_0^2 = 0$ gives

$$-n^2 - \underline{2jns} + s^2 + \underline{jn\gamma} - s\gamma + \omega_0^2 = 0 \quad \text{— (A)}$$

Hence Imaginary part is: $(-2ns + n\gamma)j = 0$

$$\Rightarrow \quad -2ns + n\gamma = 0$$

Cancelling n we get $s = \frac{\gamma}{2}$.

For the real part we take all non-imaginary parts from (A) & set to zero.

$$-n^2 + s^2 - s\gamma + \omega_0^2 = 0$$

Then if $s = \frac{\gamma}{2}$ we find

$$n^2 = \frac{\gamma^2}{4} - \frac{\gamma^2}{2} + \omega_0^2 = -\frac{\gamma^2}{4} + \omega_0^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

Thus we have shown $p = \omega_0^2 - \frac{\gamma^2}{4} + j\frac{\gamma}{2}$

What does this mean in terms of the formula for z?

Since $z = Ae^{j(pt+\alpha)}$ sub in $p = n + js$ first

$= Ae^{j(nt + jst + \alpha)}$

$= Ae^{jnt} e^{+j^2 st} e^{j\alpha}$

$= Ae^{-st} e^{j(nt+\alpha)}$

$= Ae^{-st} \{ \cos(nt+\alpha) + j\sin(nt+\alpha) \}$

So now if we take the real part:

$$x = Re\, z = Ae^{-st} \cos(nt+\alpha)$$

& subbing for n & s we get

$$x = Ae^{-\gamma t/2} \cos\left(\left[\omega_0^2 - \frac{\gamma^2}{4}\right]^{1/2} t + \alpha \right)$$

let $\omega^2 = \omega_0^2 - \dfrac{\gamma^2}{4}$ then

$$x = \underbrace{Ae^{-\gamma t/2}}_{\text{Exponential decay modifier}} \underbrace{\cos(\omega t + \alpha)}_{\text{SHM term}}$$

(note $\gamma > 0$ from its definition)

Note: both γ & ω are specified since by definition (see 6.2)

$\omega_0 = \dfrac{k}{m}$ $\quad \gamma \equiv \dfrac{b}{m}$ $\quad$ & $\quad \omega^2 = \dfrac{k^2}{m^2} - \dfrac{b^2}{4m^2}$

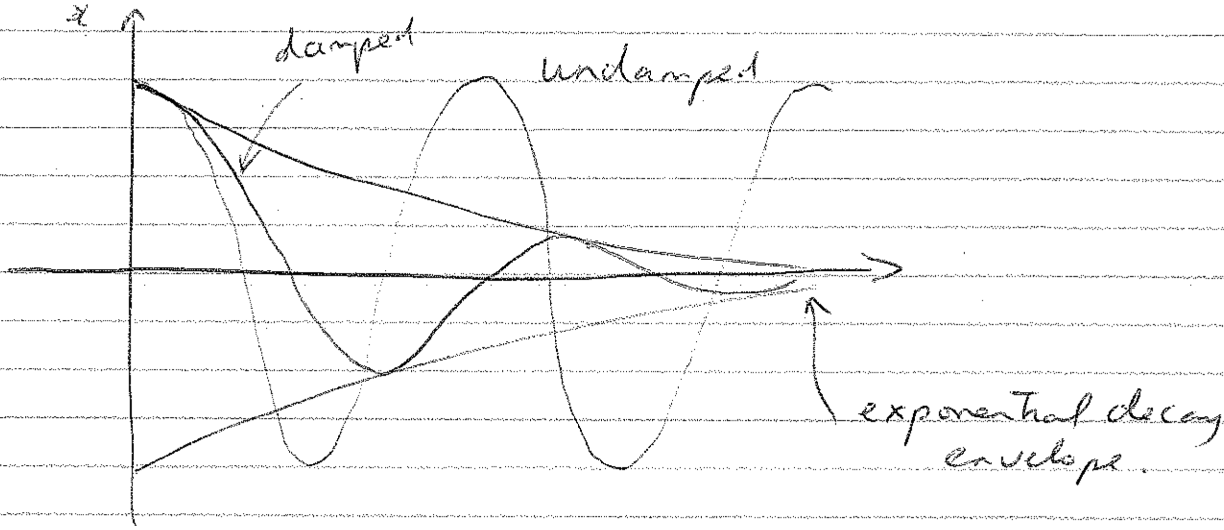
So given mass, spring constant & resistive force we can evaluate both the decay time exponent & the frequency of oscillation

Q: Does damped oscillator oscillate faster or slower than undamped?

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \frac{k^2}{m^2} - \frac{b^2}{4m}$$ so slower

makes sense

Thus the picture we get is as follows:



In the complex plane we can think of a spiral: