Free Vibrations cont.

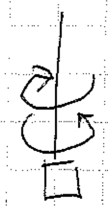
Recall: We had two approaches to SHM:

V1 (Forces) $m\ddot{x} + kx = 0$

V2 (Energy) $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E$

both equations give $\omega = \sqrt{\frac{k}{m}}$ so $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

Next let's consider torsional oscillations



applications: balance wheel of watch — See "How a watch works" on youtube about 6:30 in.
measuring viscosity
moments of inertia

A twisted object has stored energy ⇒ potential energy

In analogy with Hooke's Law let M = torque then

$$M = -c\theta$$

θ plays role of displacement

Stored energy is then

$$\int -M \, d\theta = \int^{\theta} c\theta \, d\theta = \frac{1}{2}c\theta^2$$

ie. to put energy in you have to twist against the restoring torque

Recap:

	Linear	Rotational
"Position"	x	θ
"velocity"	$\dot{x} = v$	$\dot{\theta} = \omega$
"acceleration"	$\ddot{x} = a$	$\ddot{\theta} = \alpha$
"mass"	m	inertia I
"2nd Law"	$F = ma$	$\tau = I\alpha$
"momentum"	$mv = p$	$I\omega = L$
"work"	$\int F\,dx$	$\int \tau\,d\theta$
K.E.	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$

Assume body has moment of inertia I

Kinetic energy is $\frac{1}{2}I\omega^2 = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$

Note can you calculate I for a mass m rotating on a string of length r at velocity v? Try equating the two versions of kinetic energy.

Applying conservation of energy

$$\frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2} c \theta^2 = E_{TOT}$$

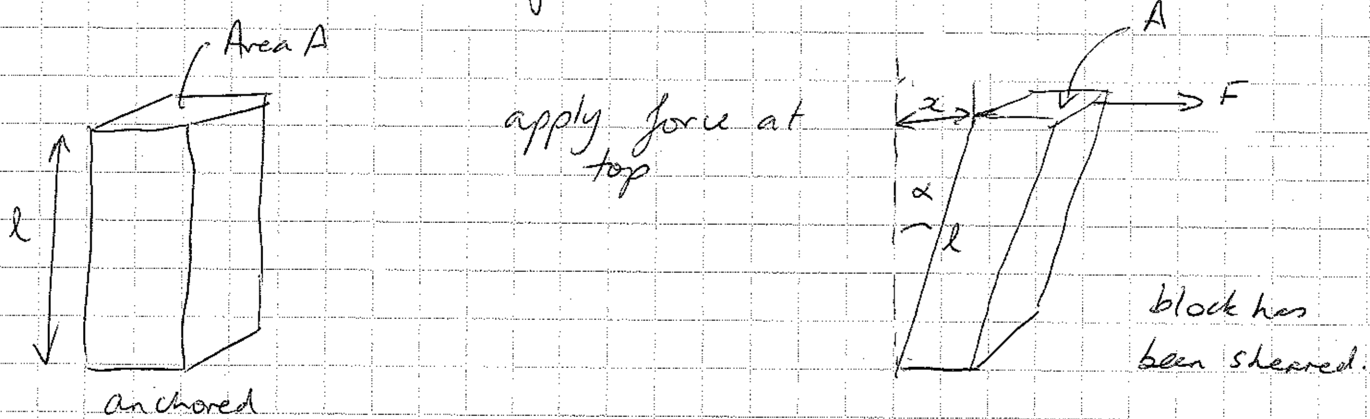
exactly equivalent to (V2) $\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = E_{TOT}$

$$\therefore \quad \omega^2 = c/I \qquad T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{c}}$$

c is a measure of the resistance to twisting, but what actually is it?

We first need to define the concept of shear.

Start with a block of material:



α is called the angle of the shear

For small angles:

$$\alpha = \frac{x}{l}$$

The more material there is in the column at fixed height, the more force will need to be applied to maintain a fixed angle of shear. We find

$$\alpha \propto \frac{F}{A}$$

There is thus a multiplier between α & $\frac{F}{A}$ ie a constant of proportionality

This constant is called the shear modulus or modulus of rigidity

It's usually written as n where

$$-\frac{F}{A} = n\alpha \qquad \Rightarrow \quad F = -nA\frac{x}{l}$$

The - sign comes because the force always wants to restore back to the original position

For small forces & displacements

$$dF = -nA\,d\alpha = -\frac{nA}{l}\,dx$$

Which looks very similar to what we got using Young's modulus & longitudinal ~~shear~~ strain:

$$dF = -\frac{AY}{l_0}\,dl$$

Note if you release the shear force the system will wobble (imagine jello!) The

$$m\ddot{x} = -\frac{nA}{l}x$$

$$\Rightarrow \quad m\ddot{x} + \frac{nA}{l}x = 0$$

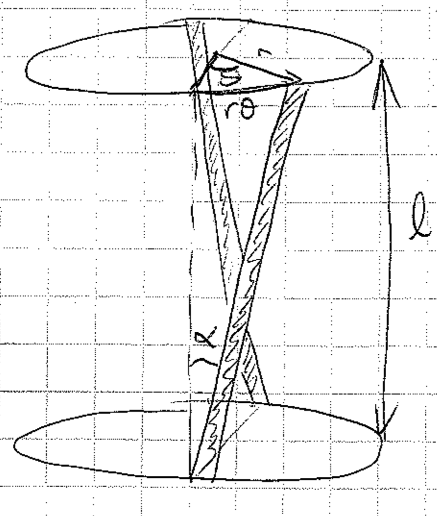
$$\Rightarrow \quad \ddot{x} + \frac{nA}{ml}x = 0$$

$$\Rightarrow \quad \omega^2 = \frac{nA}{ml} \quad \text{ie} \quad \omega = \sqrt{\frac{nA}{ml}}$$

OK! Applying to twisting:

Plates of radius r, separation l
Twisted through angle θ

Net shear angle at edge $\alpha = \frac{r\theta}{l}$

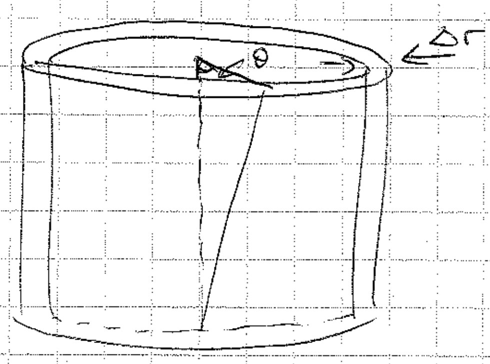


Each strip is given cross-sectional area A

Then $F = -nA\alpha = -nA\frac{r\vartheta}{l}$

The torque applied about the axis will be $rF = -nA\frac{r^2\vartheta}{l}$

If we now imagine a cylinder of thickness Δr then there are many strips all around the edge. The total net force will be the sum of all the "strips' forces" which is given by scaling up the cross section:



$$A = 2\pi r\Delta r$$

$$\therefore \Delta\tau = \Delta F.r = -n 2\pi r \Delta r \frac{r^2\vartheta}{l}$$

$$= \frac{-n2\pi r^3 \Delta r\,\vartheta}{l}$$

We can integrate this result from 0 to r to get the answer for a solid cylinder.

$$\tau = \int_0^\tau d\tau = \int_0^r -n\frac{2\pi r^3}{l}\vartheta\, dr$$

$$\Rightarrow \tau = M = -\frac{n2\pi\vartheta}{l}\int_0^r r^3\,dr = -\frac{n2\pi\vartheta}{l}\frac{r^4}{4}$$

$$\therefore M = \tau = -\frac{\pi n r^4\vartheta}{2l}$$

<u>This then tells you what c is for a cylinder!</u>

$$M = -c\vartheta = -\frac{\pi n r^4 \vartheta}{2l}$$

$$\text{So} \quad c = \frac{\pi n r^4}{2l}$$

Check out the rest of the examples in the book.