Free Vibration of Physical Systems (Chap 3)

Recall that $m\ddot{x} = -kx$ with $x = A\cos(\omega t + \alpha)$

$\ddot{x} \propto x$

$F \propto x$ is what we've used extensively.

Many physical systems have the property that

Force acting on system $\propto$ displacement

note: this may be approximate in some cases, but even so it is still a very useful form to consider

Examples:
- stretching/compression
- bending
- twisting

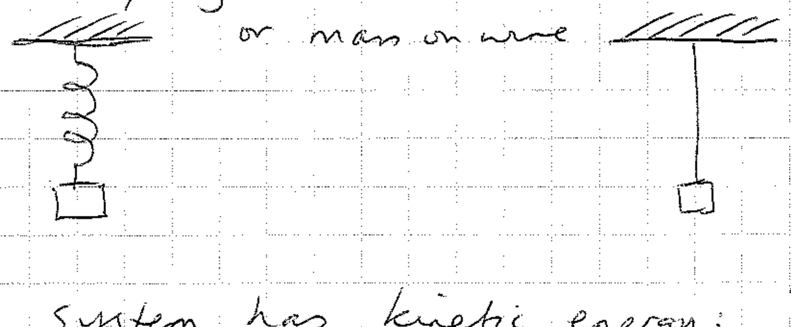
examples: spring, rod, twisting pendulum

all have "F" $\propto$ "x"

although bending & twisting won't be linear displacements & the force would be a torque

Properties we can use to classify the systems:

Start with spring: or mass on wire



- system has kinetic energy: "inertial component"
- system can store potential energy: "elastic component"

Since $F = ma$ can easily show $-kx = m\ddot{x}$ (V1)

$$\Rightarrow \ddot{x} + \frac{k}{m}x = 0$$

But this isn't the only way to analyze the problem.

We can use the conservation of energy instead.

[Aside: This is actually incredibly powerful links to the idea of "Hamiltonians" and "Lagrangians" which allow us to apply simple ideas of energy conservation to derive equations of motion that can be very complex. This is perhaps the key syllabus component of PHYS3300, Classical Mechanics.]

We know that kinetic + potential energy is conserved.

$$E_k = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2 \qquad E_p = \frac{1}{2} k x^2 \quad \left[= \int_0^x kx \, dx \right]$$

Why not $-\frac{kx^2}{2}$? Because $F = -kx$ is the restoring force ie it pushes in -ve direction. But to extend the spring & put P.E. into it you have to push in the opposite direction. Similarly if compressing the the force is -ve but so is displacement, so +ve P.E. again.

$$\text{So} \quad E_k + E_p = E_{TOT} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \qquad \text{V2}$$

Differentiate:

$$\frac{1}{\cancel{2}} . \cancel{2} m \cancel{\dot{x}} \ddot{x} + \frac{1}{\cancel{2}} . k . \cancel{2} x \cancel{\dot{x}} = 0$$

$$\Rightarrow \quad m\ddot{x} + kx = 0$$

Whenever you get equations like V1 or V2 you know you are dealing with SHM, that has the solution

$$x = A \cos(\omega t + \alpha)$$

or equivalently

$$x = Re\{z\} = Re\{A e^{j(\omega t + \alpha)}\}$$

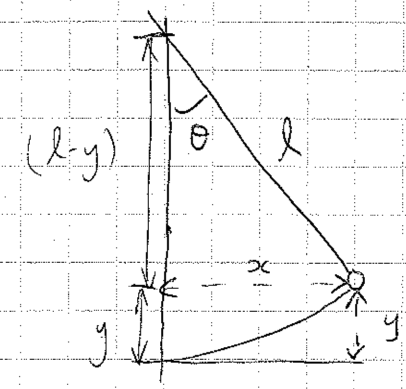
Plugging into V1 we get

$$\cancel{A}(j\omega)^2 \cancel{e^{j(\omega t + \alpha)}} + \frac{k}{m} \cancel{A e^{j(\omega t + \alpha)}} = 0$$

$$\Rightarrow \quad -\omega^2 + \frac{k}{m} = 0$$

$$\Rightarrow \quad \omega^2 = \sqrt{\frac{k}{m}} \quad \text{as expected.}$$

Simple Pendulum via conservation of energy



Potential energy comes from change in gravitational P.E. $mg \times$ height change

$\Rightarrow$ So what is y?

Need to approximate in terms of x

Since $$(l-y)^2 + x^2 = l^2$$

$$\Rightarrow \quad -2yl + y^2 + x^2 = 0$$

$$\Rightarrow \quad x^2 = 2ly - y^2 = y(2l - y)$$

Provided $2l \gg y \Rightarrow \quad x^2 \approx 2ly$

So $$y \simeq \frac{x^2}{2l}$$

We will also assume the velocity is $\dot{x}$, which means the pendulum cannot move upwards very much either (because otherwise the y-component of velocity would be significant).

Hence conservation of energy gives

$$\frac{1}{2}mv^2 + mgy = E = \frac{1}{2}m\dot{x}^2 + \frac{mg}{2l}x^2$$

$$\Rightarrow \quad \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{mg}{2l}x^2 = E$$

Differentiate

$$\frac{1}{2} \cdot 2m\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + \frac{mg}{2l}\, 2 . x . \frac{dx}{dt} = 0$$

$$\Rightarrow \quad \cancel{m}\frac{d^2x}{dt^2} + \frac{\cancel{m}g}{l}x = 0$$

$$\Rightarrow \quad \ddot{x} + \frac{g}{l}x = 0 \quad \Rightarrow \omega = \sqrt{\frac{g}{l}}$$

$$\& \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

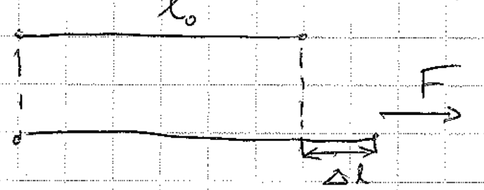
Describing Elasticity in Materials

Although we use coiled springs a lot to discuss SHM, it is easier to think about stretching a single rod or wire.

Key point: If you apply a force at one end of an object that force is felt all through the object.

If $F = -kx$ that means if we had two wires of the same length connected together the extension would be twice as long.

Definition: If a wire/rod has length l_0 but extends Δl under a force F the strain is defined by $\frac{\Delta l}{l_0}$



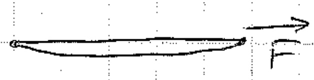
$$\text{Strain} = \frac{\Delta l}{l_0}$$

(dimensionless)

The strain is also a constant for a given type of wire and applied force.

e.g. double length of wire, Δl doubles too.

Note: if we instead put two wires in parallel:



and the apply force we have doubled the cross-sectional area of the wire & the extension will be halved (each wire need contribute only half the force now)

To maintain the same extension as the cross-sectional area is increased requires increasing the force in proportional to the area.

So $\frac{F}{A}$ = constant

Definition

$$\text{Stress} = \frac{\text{Force}}{\text{area}}$$ (same units as pressure) $N m^{-2}$ = Pascal.

= constant.

Since both strain & stress are constants (for sufficiently small strains though – things can break after all!) then

$$\frac{\text{Stress}}{\text{Strain}} = \text{constant} = \text{Young's modulus of elasticity} = \frac{F/A}{\Delta l / l} = \text{"}-Y\text{"}$$

Note that many materials break at strains between 0.1% & 1%. So the Young's modulus is not an accurate measure of overall strength. That is given by the breaking stress.

Note: if we use infintesimals we can write Y as follows:

$$-Y = \frac{dF/A}{dl/l_0}$$

$$\Rightarrow \quad dF = -\frac{AY}{l_0}\,dl \quad \text{\& integrating}$$

$$F = -\frac{AY}{l_0}x \quad \text{ie} \quad m\ddot{x} + \frac{AY}{l_0}x = 0$$

ie for a rod of length l_0, area A & made of material with Young's Modulus Y you can always calculate the force required to give a certain extension

Material:	Y / Pascal
Steel	20×10^{10}
Aluminium	6×10^{10}
Carbon nanotubes	10^{12}

Carbon nanotubes are so strong they have bee hypothesized as materials that could be used to make space elevators!

But since $$m\ddot{x} + \frac{AY}{l_0} x = 0$$

$$\Rightarrow \quad \ddot{x} + \frac{AY}{m l_0} x = 0$$

Recall $\ddot{x} + \omega^2 x = 0$ comes out of substitution with $x = A\cos(\omega t + \alpha)$

$$\Rightarrow \quad \omega = \sqrt{\frac{AY}{m l_0}}$$

This describes that natural frequency of vibration of a weight on a rod.

$$\text{Period} = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l_0 m}{AY}}$$

Lets try an example:
1 kg mass on wire length 1m, 1 mm diameter & made of steel. What is the period & frequency?

Need A, Y, m, l_0 so we have three of the four, use diameter to calculate area A

$$A = \pi \left(\frac{d}{2}\right)^2 \simeq 0.8 \times 10^{-6} \, m^{-2}$$

$$T = 2\pi \sqrt{\frac{1m \cdot 1kg}{0.8 \times 10^{-6} m^{-2} \times 20 \times 10^{10} N m^{-2}}}$$

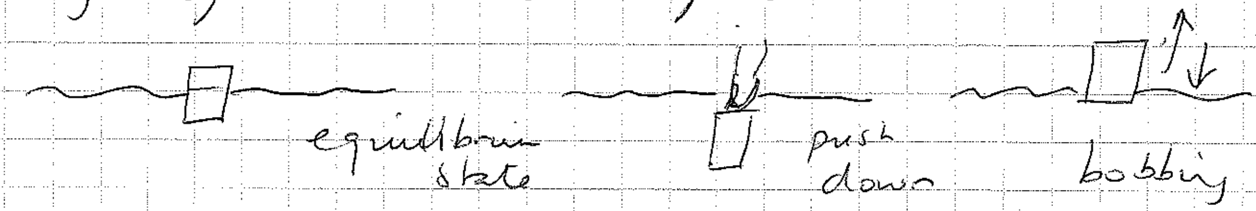
$$= 2\pi \sqrt{\frac{1}{160\,000}} = 1.6 \times 10^{-2} s$$

$$f = \nu = \frac{1}{T} \simeq 60 Hz$$

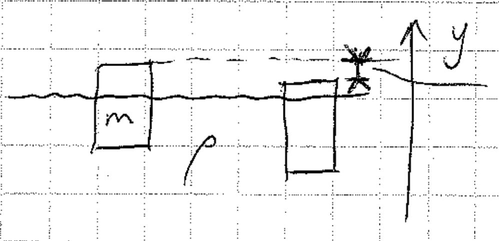
Floating objects

Believe it or not, bobbing corks are also simple harmonic oscillators!

You know that if you push down on a floating object it will respond:



Let mass of cork be m, area A
density of liquid is ρ



suppose displaced by distance y

mass of liquid displaced $= A \times y \times \rho$

But this mass wants to get "back down" to its old level, drive by gravity with acceleration $-g$.

Hence restoring force is $-gAy\rho$

Acceleration felt by cork $= ma = m\ddot{y} = -gAy\rho$

Thus $\ddot{y} + \frac{gA\rho}{m} y = 0$

So we again identify $\omega = \sqrt{\frac{gA\rho}{m}}$

& $T = 2\pi \sqrt{\frac{m}{g\rho A}}$