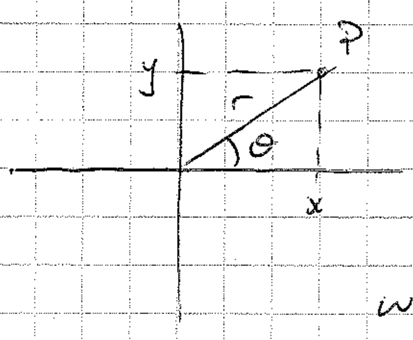
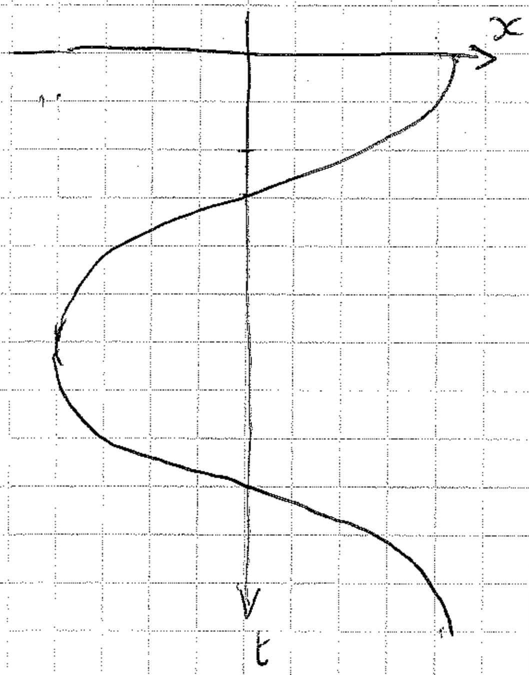
Recap: Last lecture we developed the "rotating vector" method for describing periodic motion.



Can describe position of P using
Cartesian (x, y)
Polar (r, θ)

But regardless of the words chosen we want to know the projection on to the x-axis



At the same time we showed that this motion can also be described using complex numbers.

If $|z| = \sqrt{x^2 + y^2}$ (modulus of z)

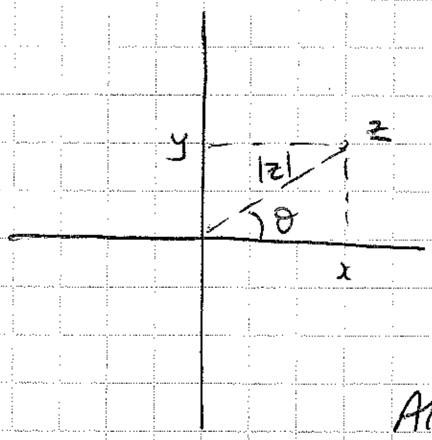
Then

"Cartesian" $z = x + jy$

"Polar" $z = |z|(\cos\theta + j\sin\theta)$

and since $e^{j\theta} = \cos\theta + j\sin\theta$

$$z = |z|e^{j\theta}$$



For complex number representation, taking the real part is equivalent to taking a projection onto the x-axis, ie the motion we're interested in

Also stated (left as homework exercise) multiplying by $e^{j\theta}$ is equivalent to rotating by θ anticlockwise.

Today: How do periodic motions add together & what do we get?

We can analyze all this using the tools we developed in the last class

Superposition of Periodic Motion

Firstly, lets consider adding motions of equal frequency but they can have different amplitudes & phases.

Let $x_1 = A_1 \cos(\omega t + \alpha_1)$ — Remember this is 1d!

$x_2 = A_2 \cos(\omega t + \alpha_2)$

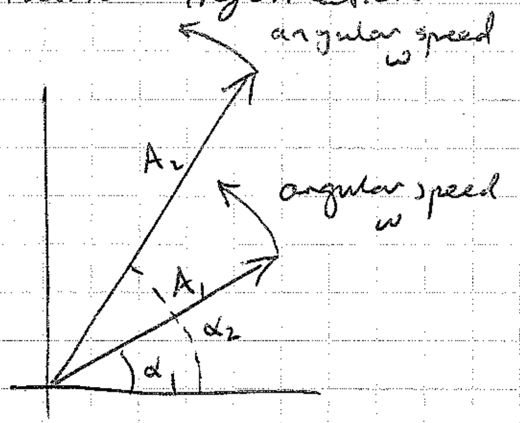
then if we set

$$x = x_1 + x_2 = A_1 \cos(\omega t + \alpha_1) + A_2 \cos(\omega t + \alpha_2)$$

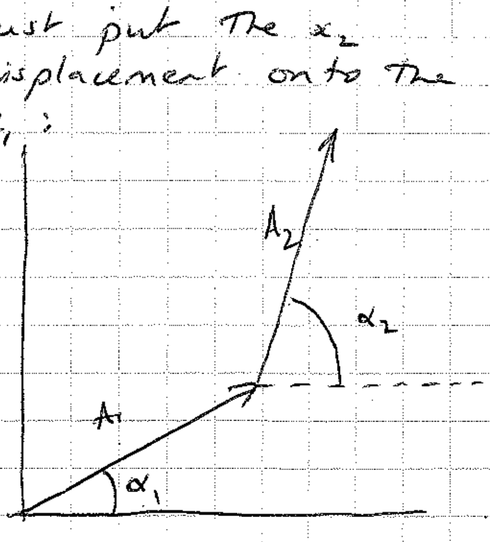
we'll show that this can actually be written

$x = A \cos(\omega t + \alpha)$ where A & α are specified by $A_1, A_2, \alpha_1, \alpha_2$

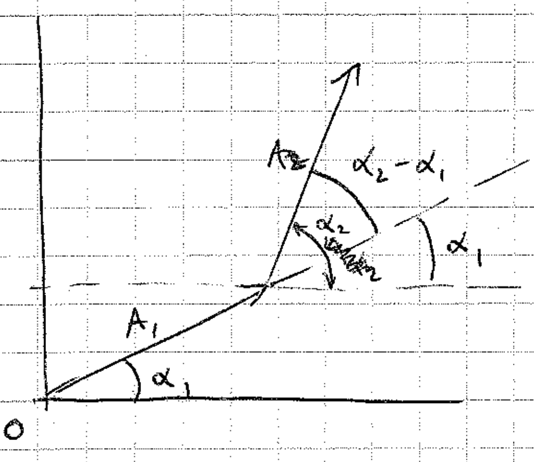
Let's use the rotating representation as this provides intuitive information!



but when added we must put the x_2 displacement onto the x_1:

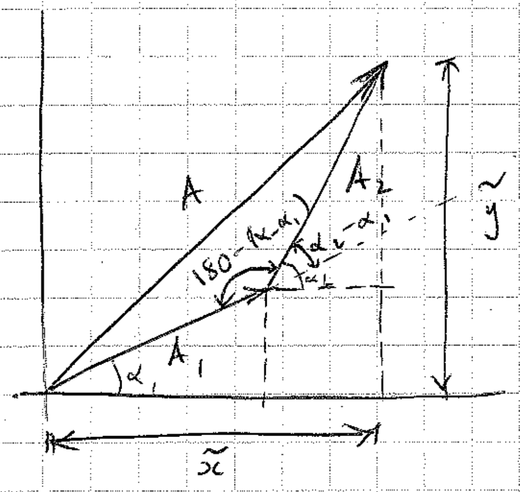


Does the angle between the two vectors change with time?

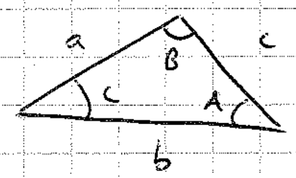


No – since the angle between the two vectors is given by $\alpha_2 - \alpha_1$

This means the distance from the origin O to the tip of A_2 must also be a constant.



Using Law of Cosines



$$c^2 = a^2 + b^2 - 2ab\cos C$$

or you can work out the position of the tip of the A vector by adding the x components to get $\tilde{x}$ & y components to get $\tilde{y}$

ie
$$\tilde{x} = A_1 \cos\alpha_1 + A_2 \cos\alpha_2$$
$$\tilde{y} = A_1 \sin\alpha_1 + A_2 \sin\alpha_2$$

The length of A is then $\sqrt{\tilde{x}^2 + \tilde{y}^2}$ so that

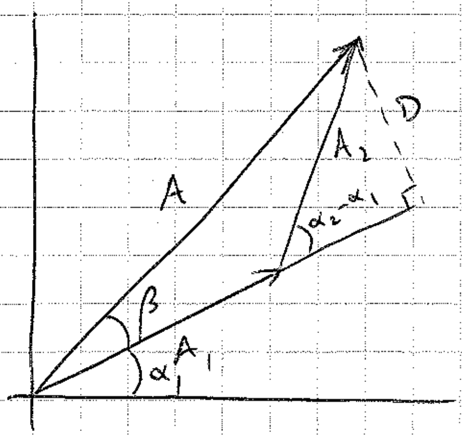
$$A^2 = A_1^2\cos^2\alpha_1 + A_2^2\cos^2\alpha_2 + 2A_1A_2\cos\alpha_1\cos\alpha_2$$
$$A_1^2\sin^2\alpha_1 + A_2^2\sin^2\alpha_2 + 2A_1A_2\sin\alpha_1\sin\alpha_2$$

Since $\cos^2 a + \sin^2 a = 1 \quad \forall a$

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\{\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2\}$$
$$= A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2 - \alpha_1)$$

since $\cos(a-b) = \cos a\cos b + \sin a\sin b$

So the length of A does not change with time either, ie the combined amplitude is fixed (as expected).



To get an equation for β just evaluate the length D for the two different triangles:

$$D = A_2 \sin(\alpha_2 - \alpha_1)$$

for the small triangle on the end

$$D = A \sin\beta$$

for the large triangle.

$$\Rightarrow \quad A \sin\beta = A_2 \sin(\alpha_2 - \alpha_1)$$

So
$$\sin\beta = \frac{A_2}{A} \sin(\alpha_2 - \alpha_1)$$

And thus given we have β we can now evaluate the total phase angle for the combined system $\alpha_1 + \beta$.

This probably feels a little bit "long winded" & can be done much faster using the complex exponential formalism.

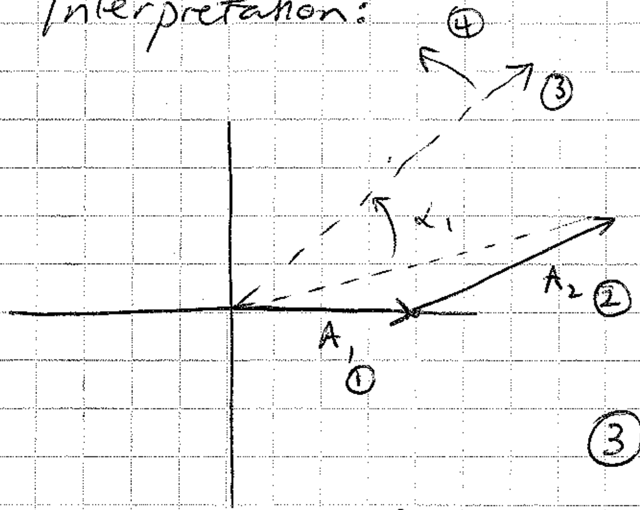
Let
$$z_1 = A_1 e^{j(\omega t + \alpha_1)}$$
$$z_2 = A_2 e^{j(\omega t + \alpha_2)}$$

Then
$$z = z_1 + z_2$$
$$= A_1 e^{j(\omega t + \alpha_1)} + A_2 e^{j(\omega t + \alpha_2)}$$
$$= A_1 e^{j\omega t} e^{j\alpha_1} + A_2 e^{j\omega t} e^{j\alpha_2}$$
$$= e^{j\omega t} e^{j\alpha_1} \left\{ A_1 + A_2 e^{j(\alpha_2 - \alpha_1)} \right\}$$

(remember $\frac{1}{e^{j\alpha_1}} = e^{-j\alpha_1}$)

So $z = e^{j(\omega t+\alpha_1)}\left\{A_1 + A_2 e^{j(\alpha_2-\alpha_1)}\right\}$

Interpretation:



① First offset along x-axis by A_1

② From end of A_1 add A_2 but rotate through angle $\alpha_2-\alpha_1$

③ Then rotate by angle α_1

④ System will the evolve with time with angular speed ω.

If the two amplitudes are equal the the result is particularly simple:

Using double angle formula $\cos 2A = \cos^2 A - \sin^2 A$

$\Rightarrow \cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$

& setting $A_2 = A_1$ then gives for the formula for A

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2-\alpha_1) = 2A_1^2 + 2A_1^2\cos(\alpha_2-\alpha_1)$$

$$= 2A_1^2\left\{1+\cos(\alpha_2-\alpha_1)\right\} = 2A_1^2\left\{1+\cos^2\left(\frac{\alpha_2-\alpha_1}{2}\right) - \sin^2\left(\frac{\alpha_2-\alpha_1}{2}\right)\right\}$$

$$= 2A_1^2\left\{2\cos^2\left(\frac{\alpha_2-\alpha_1}{2}\right)\right\}$$

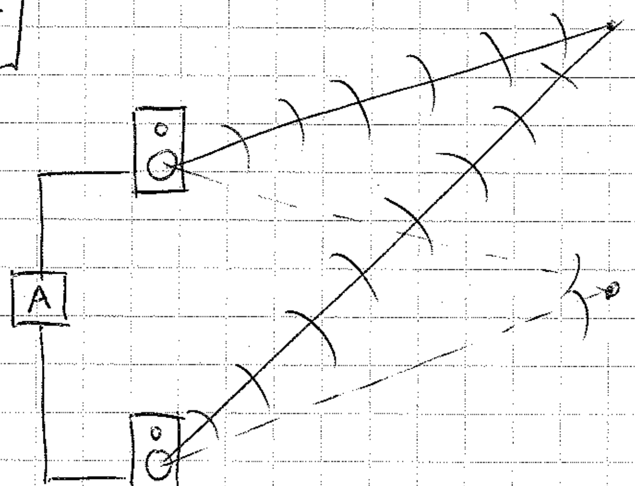
$\therefore \quad A = 2A_1 \cos\left(\frac{\alpha_2-\alpha_1}{2}\right)$

and using $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$ can show

$\beta = \frac{\alpha_2-\alpha_1}{2}$ (can also read off by inspection)

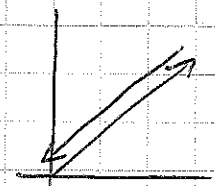
A similar situation happens when two wave sources combine together - This is <u>interference</u>.

[Comb filter demo]

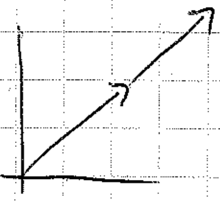


Arrival out of phase by $n\pi$

"Destructive interference"



Arrival in phase or multiples of 2π

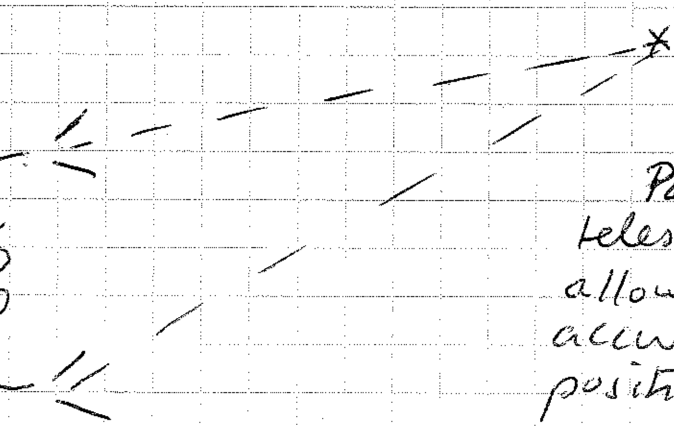


"Constructive interference"

At other places $\alpha_2 - \alpha_1$ will vary

Note - Reverse application of this idea leads to the idea of interferometry. Very important for radio telescope design (& other wavelengths)

Constructive interference only when telescopes point in same direction



~~Positions~~ Directions that telescopes point at allow you to very accurately pinpoint positions.

Vibrations of <u>different</u> frequencies.

If $\omega_1 \neq \omega_2$ Then phase angle is constantly changing.

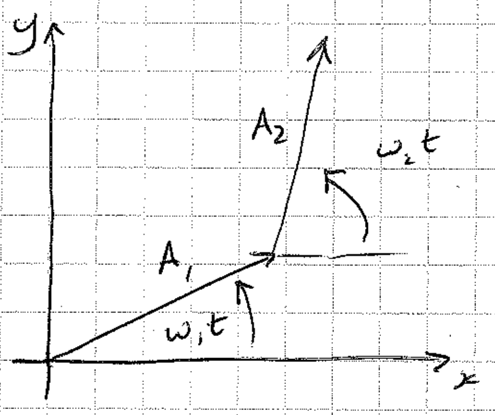
This means the initial phase angles are not very important, so to make the math easier set them to zero.

Let

$$x_1 = A_1 \cos \omega_1 t$$

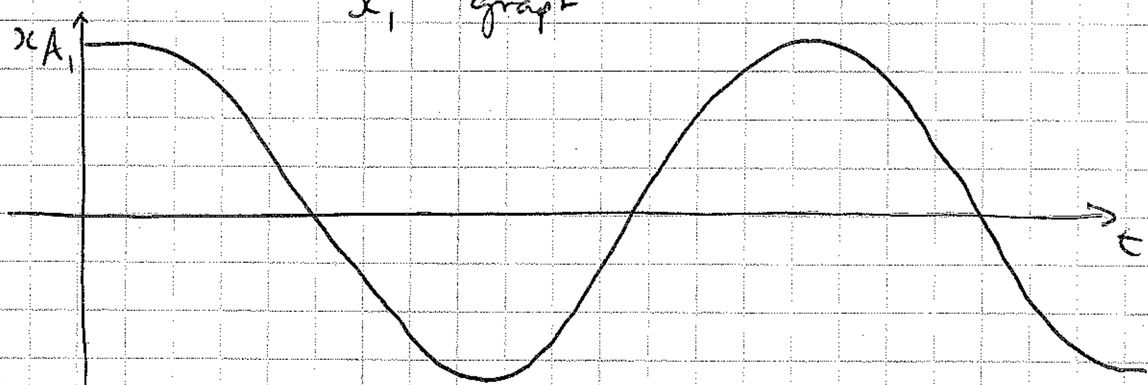
$$x_2 = A_2 \cos \omega_2 t$$

Think about adding as before:

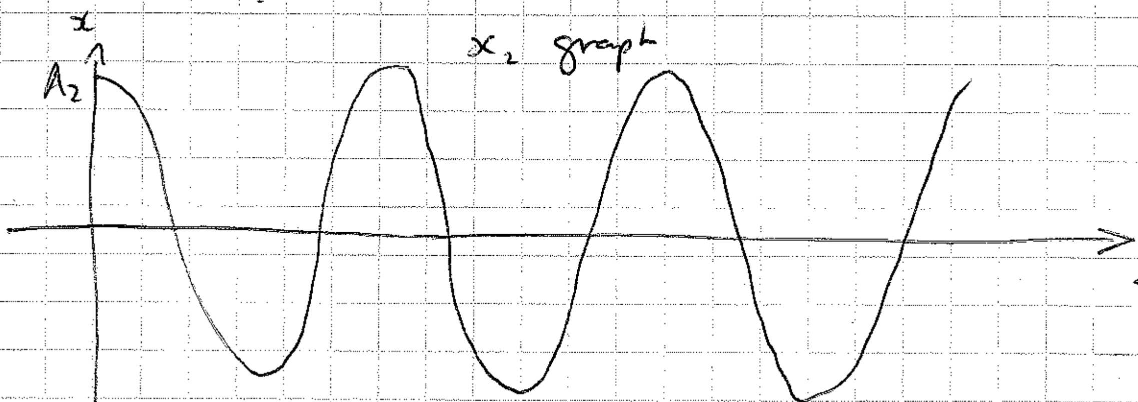


but now have different frequencies

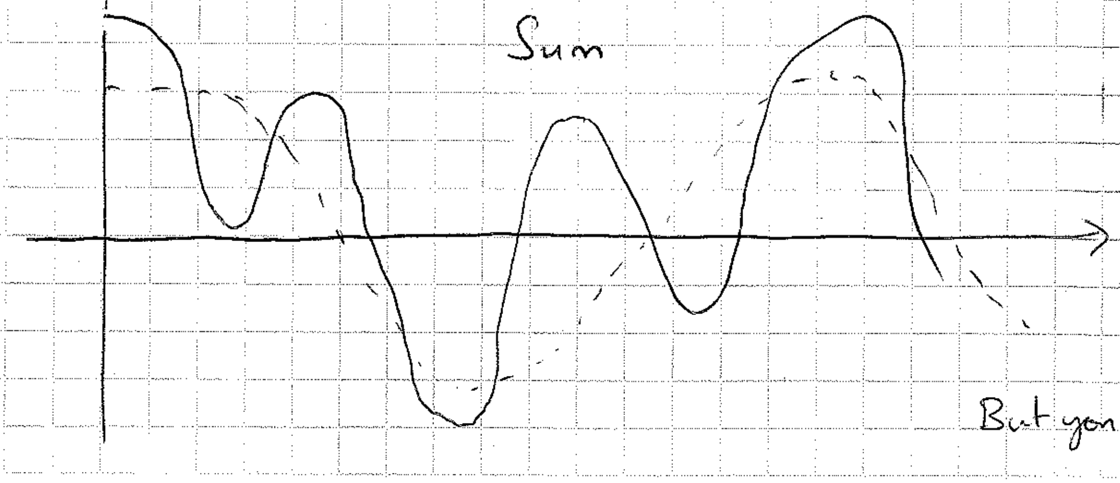
x_1 graph



x_2 graph



Sum



poorly drawn!

But you get the idea...

Beats:

[Tuning fork demo.]

Happens when two frequencies are very close ie $\omega_1 \approx \omega_2$

let $\omega_1 = \omega + \Delta\omega$ where $\Delta\omega$ is very
$\omega_2 = \omega - \Delta\omega$ small relative to ω so that $\omega_1 \approx \omega_2$

Then $\omega_1 + \omega_2 = 2\omega \Rightarrow \omega = \dfrac{\omega_1 + \omega_2}{2}$

$\omega_1 - \omega_2 = 2\Delta\omega \Rightarrow \Delta\omega = \dfrac{\omega_1 - \omega_2}{2}$

Then the sum of two equal amplitude (to make math easier) SHMs is

$$x = x_1 + x_2 = A\left[\cos\omega_1 t + \cos\omega_2 t\right]$$

$$= A\left[\cos(\omega + \Delta\omega)t + \cos(\omega - \Delta\omega)t\right]$$

Using $\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= A\left[\cos(\omega t)\cos(\Delta\omega t) - \cancel{\sin(\omega t)\sin(\Delta\omega t)} + \cos(\omega t)\cos(\Delta\omega t) + \cancel{\sin(\omega t)\sin(\Delta\omega t)}\right]$$

$$= 2A\cos(\omega t)\cos(\Delta\omega t)$$

$$= 2A\cos\left[\left(\frac{\omega_1+\omega_2}{2}\right)t\right]\underbrace{\cos\left[\left(\frac{\omega_1-\omega_2}{2}\right)t\right]}_{\text{envelope}}$$

"sound" at average frequency