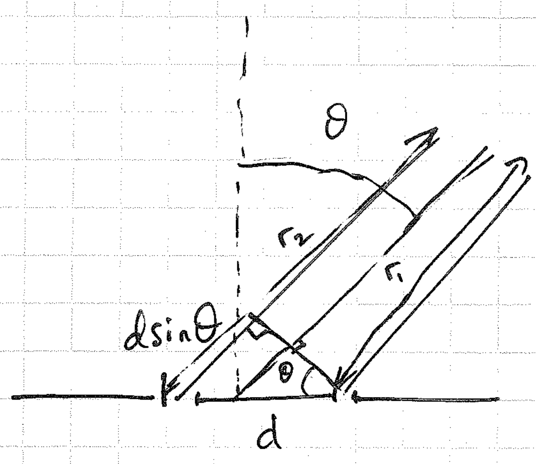
Application of the Huygen-Fresnel Principle

Double-slit Interference (Shows up in quantum too!)



Two slits are the sources of spherical wavefronts

Two wavefronts are both in-phase.

Obviously the resultant interference pattern will be a super-position of the two wavefronts.

At some positions the wavefronts will be exactly in phase - implies maximal constructive interference.

At other positions the wavefronts will be 180° out of phase - implies destructive interference

(There are of course different phase difference)

Look at wave tank demo of double-slit - see lines where there is little or no amplitude

$\Rightarrow$ "nodal lines"

Q: Where do the nodal lines occur?

A: Whenever the path length difference $= \frac{\lambda}{2}$
ie when $d \sin \theta = \frac{\lambda}{2}$

To analyse further let's write out the two wavefronts and suppe-pose them.

let $y_1 = A_0 \cos\left[\omega\left(t - \frac{r_1}{v}\right)\right]$

here r_1 describes the radial distance to a point p, so this gives us a hemispherical wave as we want.

Note: In fact the amplitude from a point source will decay with distance, but we'll ignore that for now.

To the same point p to the second slit the distance is r_2, so we can write the wavefront y_2 at that point as

$$y_2 = A_0 \cos\left[\omega\left(t - \frac{r_2}{v}\right)\right]$$

So now we can add to get the total disturbance at p:

$$y_p = y_1 + y_2$$

$$= A_0\left[\cos\left[\omega\left(t - \frac{r_1}{v}\right)\right] + \cos\left[\omega\left(t - \frac{r_2}{v}\right)\right]\right]$$

use the good old trig substitution

$$\cos\theta\cos\phi = \frac{1}{2}\left[\cos(\theta+\phi) + \cos(\theta-\phi)\right]$$

$$= 2A_0 \cos\omega t \cos\left[\frac{\omega}{2v}(r_2 - r_1)\right]$$

or since $\lambda = \frac{v}{\nu} = \frac{2\pi v}{\omega}$

$$= 2A_0 \underbrace{\cos\omega t}_{\text{time dependence}} \underbrace{\cos\left[\frac{\pi(r_2 - r_1)}{\lambda}\right]}_{\text{Spatial dependence}} \qquad (*)$$

Finding the nodal points means finding all situations where $y_P = 0$.

Ignore time dependence term & focus on the spatial part. Then need

$$\cos\left[\frac{\pi(r_2 - r_1)}{\lambda}\right] = 0$$

$$\Rightarrow \quad \frac{\pi(r_2 - r_1)}{\lambda} = \left(n + \tfrac{1}{2}\right)\pi \quad \text{or} \quad (2n+1)\frac{\pi}{2} \qquad n = 0, 1, 2, 3,$$

$$\Rightarrow \quad r_2 - r_1 = \left(n + \tfrac{1}{2}\right)\lambda$$

But since $r_2 - r_1 = d\sin\theta$

$$\Rightarrow \quad \sin\theta = \frac{\lambda}{d}\left(n + \tfrac{1}{2}\right)$$

What about maxima of y_P?

Then we have $\cos\left[\frac{\pi(r_2 - r_1)}{\lambda}\right] = 1$

$$\text{ie} \quad \frac{\pi(r_2 - r_1)}{\lambda} = n\pi \qquad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \quad r_2 - r_1 = n\lambda$$

$$\therefore \quad \sin\theta = \frac{n\lambda}{d}$$

What is the amplitude as a function of θ?
Using $d\sin\theta = r_2 - r_1$ we can substitute into (*)

$$A(\theta) = 2A_0 \cos\left(\frac{\pi d \sin\theta}{\lambda}\right) \qquad \text{ignoring time part}$$

So the amplitude depends upon the direction.

What happens when $d \gg \lambda$?

In this situation we can consider small angles around the central peak. That means $\sin\theta_n \rightarrow \theta_n$

Then angular separation between maxima becomes

$$\sin\theta \sim \theta = \frac{n\lambda}{d}$$

[ie separated by $\frac{\lambda}{d}$]

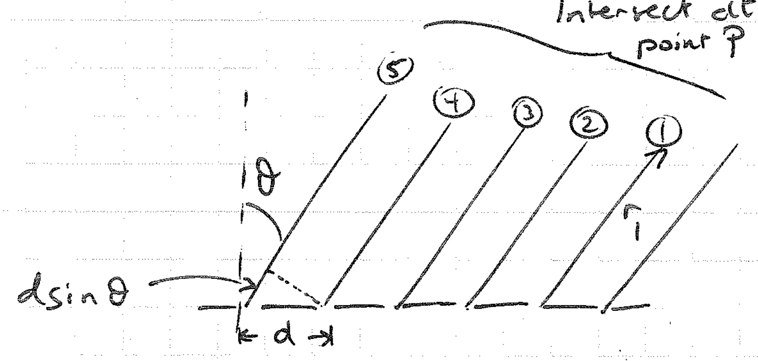
Multiple-Slit Interference – Diffraction gratings

Very important concept in astronomy instrumentation. They disperse light via diffraction rather than refraction.

Fun examples – CD surface
"diffraction glasses"

Diffraction gratings are used rather than prisms because they can be easily tailored to the desired wavelength and they are comparatively cheap. Prisms have disadvantages in that they may not transmit certain wavelengths (e.g. UV) and they may not be able to separate wavelengths very strongly ("low dispersion").

Setting the problem up:



Q: What is the path difference between slits?

A: $d \sin\theta$

Q: What arrival time difference?

A: $\frac{d\sin\theta}{v}$

Q: What phase difference does this correspond to?

A: $2\pi \times \left(\frac{d}{v}\sin\theta\right)/T = \frac{\omega d}{v}\sin\theta = \frac{2\pi d}{\lambda}\sin\theta = \delta$

Contribution to point P by the first slit is

$$y_1 = A_0 \cos\left[\omega\left(t - \frac{r_1}{v}\right)\right] = A_0 \cos\left(\omega t - \frac{\omega r_1}{v}\right)$$

where r_1 is the distance to point P.

Let $\frac{\omega r_1}{v} = \frac{2\pi r_1}{\lambda} = \varphi_1$ to simplify notation
(+ it gives the phase difference to the point P)

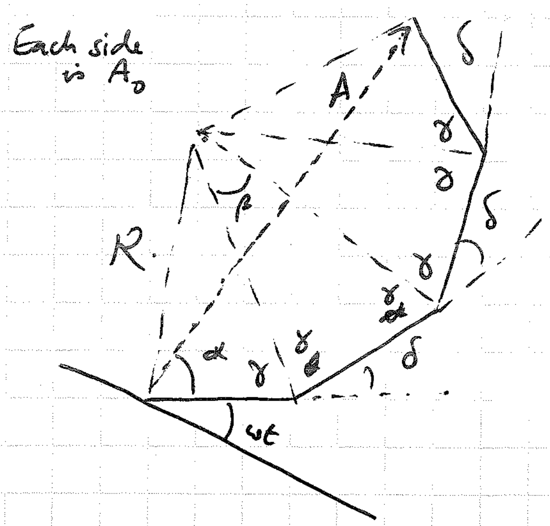
Then adding the contributions from all the other slits we have:

$$y_P(t) = A_0 \cos(\omega t - \varphi_1) + A_0 \cos(\omega t - \varphi_1 - \delta)$$
$$+ A_0 \cos(\omega t - \varphi_1 - 2\delta) + \ldots \quad \text{to } N \text{ terms if } N \text{ slits}$$

Q: How do we evaluate this superposition?

A: We've actually seen this already in Chap 2! p.27

Same amplitude but different phases inscribes inside a circle

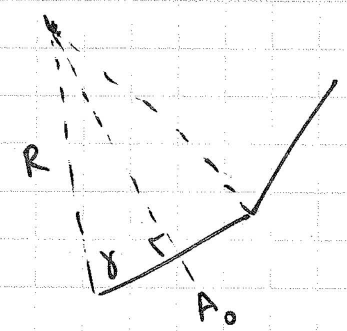


Notice $\gamma + \gamma + \delta = \pi$ by looking at the sum of angles at the edges.

But inside triangle $\gamma + \gamma + \beta = \pi$

$\therefore \beta = \delta$

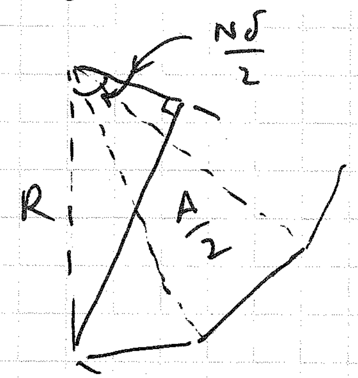
If we cut one triangle in half



$$\frac{A_0}{2R} = \sin\left(\frac{\beta}{2}\right) = \sin\left(\frac{\delta}{2}\right)$$

$$\Rightarrow \quad A_0 = 2R\sin\left(\frac{\delta}{2}\right)$$

For the triangle subtended by A & the two sides length R the interior angle is $N\delta$. To use a right angle triangle we again cut the angle in half.



$$\Rightarrow \quad \sin\left(\frac{N\delta}{2}\right) = A/2R$$

$$\Rightarrow \quad A = 2R\sin\left(\frac{N\delta}{2}\right)$$

$$\therefore \quad A = A_0 \frac{\sin(N\delta/2)}{\sin(\delta/2)}$$

The question now is how does A vary with δ?
(which relates directly to θ)

Case 1: If $\delta = 0$

$A = NA_0$ This is the biggest possible amplitude

Case 2:
When is $A = 0$? When the polygon is <u>closed</u>.

i.e. The side A becomes smaller & smaller as the polygon gets closer & closer to being closed.

This will happen when $N\delta = 2\pi$ (or multiples)

$$\Rightarrow \delta = \frac{2\pi n}{N} \quad \left(\text{ie } \frac{2\pi}{N}, \frac{4\pi}{N}, \frac{6\pi}{N} \text{ etc}\right)$$

Case 3: In between zeros there will be smaller maxima called subsidiary maxima. Their amplitudes are considerably lower than the principal maxima.

e.g. for $N=8$