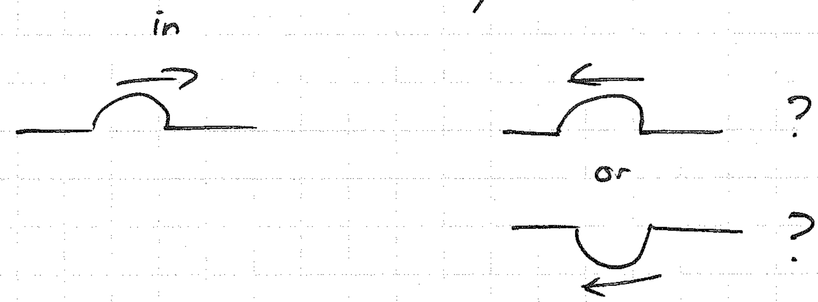
Boundary Effects & Interference

Thus far we have studied waves that travel essentially unimpeded (with the exception of the standing wave example).

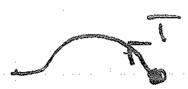
However, we'll now look at what happens when a travelling wave encounters an obstacle, barrier, or moves into a different medium.

As a first example consider reflections from the ends of springs.

Q: For a slinky with an anchored end does it reflect back in phase or out of phase? ie



A: Think about the end point. Tension in the slinky is pulling the end point upwards:

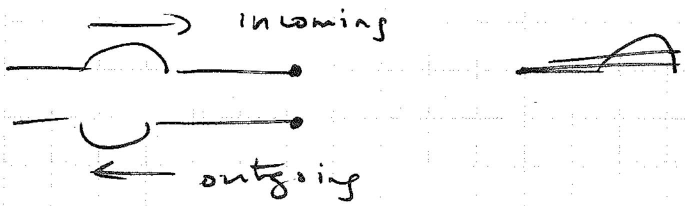


This must be balanced by an equal & opposite force at the end to remain stationary.

[- Try it yourself by holding the end - you can feel the tension as the pulse hits.]

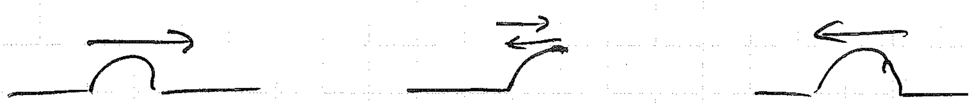
So the end point exerts an opposite force which then starts to propagate as well!

The net result is



Q: If we let the end point be free what happens?

A: With the end point no longer anchored it moves up in response to the incoming pulse:



So the new pulse reflects back the same way as the incoming pulse.

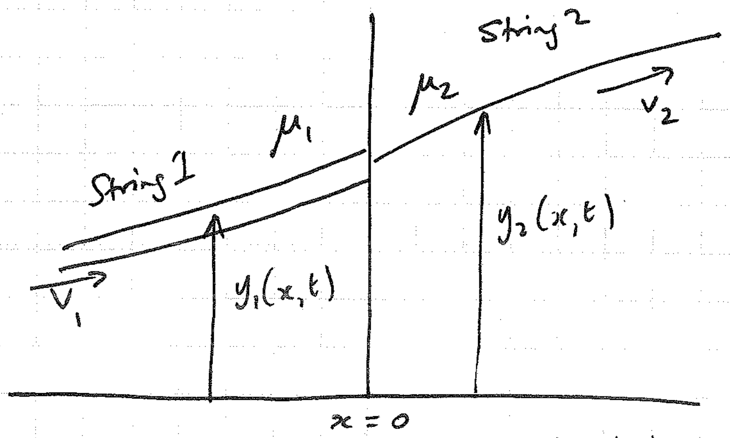
Q: For water waves hitting a pier which of the above cases applies?
Similarly for sound waves hitting a wall?

Partial transmission & reflection

When the material the wave is propagating through changes its properties (most notably mass density) there will be an impact on any propagating waves.

Note, this also applies to light waves transitting from air $\rightarrow$ glass (for example)
Or water waves moving from deep to shallow water.

For an example we'll consider a string where the mass density changes.



In this case we'll find there is partial reflection & transmission at the interface.

Suppose the incoming (pulse) wave is denoted f_1, there will also be a reflected pulse we'll call g_1, & a transmitted pulse f_2.

On the left of the boundary we have

$$y_1(x,t) = \underbrace{f_1\left(t - \frac{x}{v_1}\right)}_{\text{pulse/wave travelling to right}} + \underbrace{g_1\left(t + \frac{x}{v_1}\right)}_{\text{travelling to left}}$$

note same velocities here

on the right

$$y_2(x,t) = \underbrace{f_2\left(t - \frac{x}{v_2}\right)}_{\text{travelling rightward}}$$

note different velocity here.

At $x=0$ the two strings are at the same point

ie $\quad y_1(0,t) = y_2(0,t)$

$$\Rightarrow \quad f_1(t) + g_1(t) = f_2(t) \qquad \text{—— (1)}$$

Q: What other property do we need to match at $x=0$?

A: The derivatives also must match. If that wasn't the case there would be big accelerations at $x=0$!

$$\therefore \quad \frac{\partial y_1}{\partial x}(0,t) = \frac{\partial y_2}{\partial x}(0,t)$$

$$\Rightarrow \quad -\frac{1}{v_1} f_1'(t) + \frac{1}{v_1} g_1'(t) = -\frac{1}{v_2} f_2'(t)$$

We can integrate this wrt to x to get

$$-\frac{1}{v_1} f_1(t) + \frac{1}{v_1} g_1(t) = -\frac{1}{v_2} f_2(t)$$

$$\Rightarrow \quad v_2 f_1(t) - v_2 g_1(t) = v_1 f_2(t) \quad \text{———— (2)}$$

Substitute for f_2 in (2) using (1) to get

$$v_2 f_1(t) - v_2 g_1(t) = v_1 f_1(t) + v_1 g_1(t)$$

$$\Rightarrow \quad f_1(t)(v_2 - v_1) = g_1(t)(v_1 + v_2)$$

$$\Rightarrow \quad g_1(t) = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) f_1(t) \quad \text{———— (3)}$$

———— tells you about reflected pulse at interface

Then use this to substitute into (1) to get f_2 as a function of f_1:

$$f_1(t) + \frac{v_2 - v_1}{v_1 + v_2} f_1(t) = f_2(t)$$

$$\Rightarrow \quad f_1(t)\left(\frac{2v_2}{v_1 + v_2}\right) = f_2(t) \quad \text{———— (4)}$$

———— tells you about the transmitted pulse. at interface

Thus far we've looked at $x=0$ alone, for any t. Now lets extend the analysis to include time variation.

Let $t' = t + \Delta t$ but what value should we choose for Δt? Let's choose $\Delta t = -\frac{x}{v_1}$ as then

the value in $f_1(t+\Delta t)$ is $f_1\left(t - \frac{x}{v_1}\right)$ which looks

like what we started with!

Obviously x corresponds to distance on the string related to the distance the pulse travels.

Hence from (3)

$$g_1(t+\Delta t) = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) f_1(t+\Delta t)$$

becomes

$$g_1\left(t - \frac{x}{v_1}\right) = \frac{v_2 - v_1}{v_1 + v_2} f_1\left(t - \frac{x}{v_1}\right)$$

but $g_1\left(t - \frac{x}{v}\right)$ is not what we had! g_1 was travelling to the left!

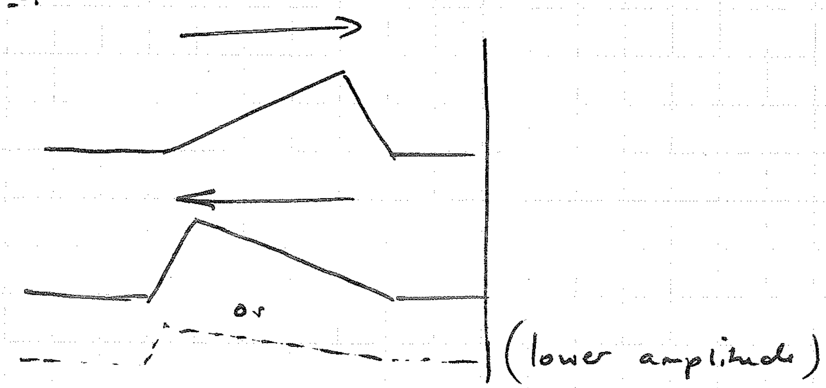
Interpretation

$$g_1\left(t - \frac{x}{v_1}\right) \equiv g_1\left(t + \left(\frac{-x}{v_1}\right)\right)$$

travelling left

but shape flipped horizontally!

i.e.



What about the amplitude?

$\frac{v_2 - v_1}{v_1 + v_2}$ means <u>amplitude of g_1 is reduced compared to f_1</u>

+ if $v_2 < v_1$ then the <u>amplitude will be flipped upside down!</u>

For the transmitted pulse we get

$$f_2\left(t-\frac{x}{v_1}\right) = \frac{2v_2}{v_1+v_2} f_1\left(t-\frac{x}{v_1}\right)$$

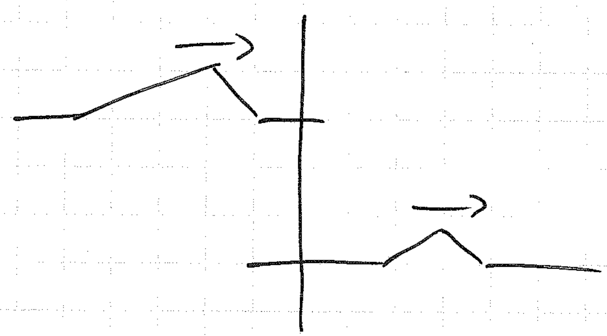
$$\Rightarrow \quad f_2\left(t - \frac{v_2}{v_1}\frac{x}{v_2}\right) = \frac{2v_2}{v_1+v_2} f_1\left(t-\frac{x}{v_1}\right)$$

transmitted

but x-profile is either compressed or stretched

Can describe as propagating at v_2

amplitude is adjusted as well



Let's check these results relative to our consideration of the end points.

(a) String tied at $x=0$ $\Rightarrow$ $\mu_2 \to \infty$ $\Rightarrow$ $v_2 = \sqrt{\frac{T}{\mu}} = 0$

$$\Rightarrow \quad g_1\left(t-\frac{x}{v_1}\right) = \frac{0-v_1}{v_1+0} f_1\left(t-\frac{x}{v_1}\right)$$

$$\Rightarrow \quad g_1\left(t+-\frac{x}{v_1}\right) = -f_1\left(t-\frac{x}{v_1}\right)$$

$\therefore$ perfect reflection with a reversal of the amplitude ✓ ☺

(b) Now let end move i.e. mass of second string is zero $\Rightarrow$ v_2 is very large!

$$\Rightarrow \quad g_1\left(t-\frac{x}{v_1}\right) = \frac{v_2}{v_2} f_1\left(t-\frac{x}{v_1}\right)$$

$$\Rightarrow \quad g_1\left(t+-\frac{x}{v_1}\right) = f_1\left(t-\frac{x}{v_1}\right)$$

So we get perfect reflection with no change in direction of the amplitude

(c) We can also look at $v_1 = v_2$ ie no change in μ!

$\Rightarrow \quad g_1(t + -\frac{x}{v_1}) = 0$ as $v_2 - v_1$ cancels.

So wave continues as expected.

So it seems to work!