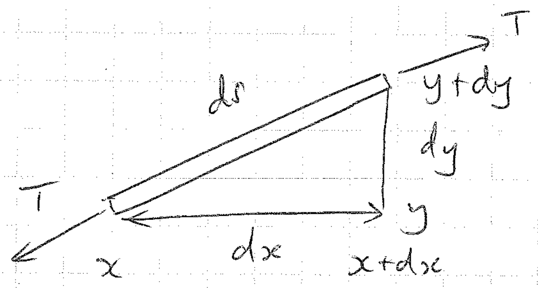
Energy in a wave (a mechanical one!)

As wave travels through medium the particles of the medium are perturbed $\Rightarrow$ energy is carried by the wave.

Q: How can we calculate this energy?

A: Let's consider the string example first, but key point is to consider both the potential & kinetic energies.



Element of length dx initially is stretched to length ds. ds is going to be related to the mass of the segment (via dx) & also the amount of stretch we have

First part: evaluate the kinetic energy:

$$K.E. = \frac{1}{2} m v^2 \qquad \text{so need mass + velocity}$$

$$= \frac{1}{2} (\mu . dx) \left(\frac{\partial y}{\partial t}\right)^2$$

dx because ds is the stretched length.

but this is the K.E. for an infinitesimal piece, so we need to normalize, thus divide by dx to get the K.E. per unit length (K.E. density if you like)

$$\frac{K.E.}{dx} = \frac{\frac{1}{2} (\mu . dx) \left(\frac{\partial y}{\partial t}\right)^2}{dx} = \frac{1}{2} \mu \left(\frac{\partial y}{\partial t}\right)^2$$

The P.E. comes from the work done in stretching the string. Thus far we've ignored the stretching but it isn't hard to evaluate.

Since the tension is the measure of the force, The- the work done in stretching = for a x distance

$\Rightarrow \quad P.E. = T\,(ds - dx)$

$\quad\quad$ i.e. subtract off the initial length

So now need ds:

Apply Pythagoras:

$$ds^2 = dx^2 + dy^2$$

$$\Rightarrow \quad ds = dx\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{1/2}$$

partial since $y(x, t)$

$$= dx\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \ldots\right)$$

applying Taylor expansion for small $\left(\frac{\partial y}{\partial x}\right)^2$

$$\therefore \quad ds - dx = dx\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \ldots\right) - dx$$

$$\simeq \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Thus $\quad P.E. = T\,(ds - dx)$

$$= T \cdot \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

So again need to normalize by dx to get P.E. per unit length (or P.E. density)

$$\frac{P.E.}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

and hence the total energy density is

$$\frac{K.E.}{dx} + \frac{P.E.}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \quad (\ast\ast\ast)$$

Let's try evaluating this for a sine wave.

Given a travelling wave train:

$$y(x,t) = A\sin\left[\frac{2\pi}{\lambda}(vt - x)\right]$$

Since one wavelength for the entire wave, write
$V = \lambda\nu$

$$\Rightarrow \quad y(x,t) = A\sin\left[\frac{2\pi\nu}{V}(Vt - x)\right]$$
$$= A\sin\left[2\pi\nu\left(t - \frac{x}{V}\right)\right]$$

Now need to plug into (***), for K.E. first

$$\frac{\partial y}{\partial t} = \underbrace{2\pi\nu A}_{\text{must be max velocity in } y \text{ direction}} \cos\left[2\pi\nu\left(t - \frac{x}{V}\right)\right]$$

$$= u_0 \cos\left[2\pi\nu\left(t - \frac{x}{V}\right)\right]$$

at $t=0$ we find

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = u_0 \cos\left[-\frac{2\pi\nu x}{V}\right] = u_0\cos\left[-\frac{2\pi x}{\lambda}\right]$$

$$= u_0\cos\left[\frac{2\pi x}{\lambda}\right]$$ since cos is symmetric

So $$\frac{K.E.}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 \quad \left(= \frac{dk}{dx}\right)$$

$$= \frac{1}{2}\mu u_0^2 \cos^2\left(\frac{2\pi x}{\lambda}\right)$$

Now this is the K.E. density ~~per unit~~ so to evaluate energy in one wavelength we need to integrate:

$$\text{K.E. of one } \lambda = \int_0^\lambda \frac{dk}{dx}\,dx = \frac{1}{2}\mu u_0^2 \int_0^\lambda \cos^2\left(\frac{2\pi x}{\lambda}\right) dx$$

Now need to evaluate

$$\int_0^\lambda \cos^2\left(\frac{2\pi x}{\lambda}\right) dx \qquad \text{use} \quad \cos^2 a = \frac{1}{2}(1+\cos 2a)$$

$$\text{let} \quad a = \frac{2\pi x}{\lambda} \Rightarrow da = \frac{2\pi}{\lambda} dx$$

$$= \int_0^{x=\lambda} \cos^2 a \,.\, \frac{\lambda}{2\pi}\, da$$

$$= \int_0^{a=2\pi} \frac{\lambda}{2\pi} \frac{1}{2}\left\{1 + \cos 2a\right\} da$$

$$= \frac{\lambda}{4\pi} \int_0^{a=2\pi} (1 + \cancel{\cos 2a})\, da$$

$$= \frac{\lambda}{4\pi}\Big[a\Big]_0^{2\pi} = \frac{\lambda}{2}$$

$$\therefore\ K = \frac{1}{2}\mu u_0^2 \int_0^\lambda \cos^2\left(\frac{2\pi x}{\lambda}\right) dx = \frac{1}{4}\mu u_0^2 \lambda$$

and subbing for $u_0 = 2\pi\nu A$

$$K = \frac{1}{\cancel{4}}\mu \cancel{4}\pi^2\nu^2 A^2\lambda = \pi^2\mu A^2\lambda\nu^2$$

Next calculate the P.E. term:

$$\frac{\partial y}{\partial x} = -\frac{2\pi\nu A}{v}\cos\left[\frac{2\pi\nu}{v}(vt - x)\right]$$

$$= -\frac{2\pi\nu A}{v}\cos\left[2\pi\nu\left(t - \frac{x}{v}\right)\right]$$

now at $t=0$

$$\left.\frac{\partial y}{\partial x}\right|_{t=0} = -\frac{2\pi\nu A}{v}\cos\left(-\frac{2\pi\nu x}{v}\right) = -\frac{2\pi\nu A}{v}\cos\left(\frac{2\pi x}{\lambda}\right)$$

Hence the PE density is

$$\frac{dU}{dx} = \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2$$

$$= \frac{1}{2} T \left(\frac{2\pi A}{\lambda}\right)^2 \cos^2\left(\frac{2\pi x}{\lambda}\right)$$

$$= \frac{2\pi^2 A^2 T}{\lambda^2} \cos^2\left(\frac{2\pi x}{\lambda}\right)$$

To get the energy in one cycle integrate over the wavelength:

$$U = \int_0^\lambda \frac{dU}{dx}\,dx = \frac{2\pi^2 A^2 T}{\lambda^2} \int_0^\lambda \cos^2\left(\frac{2\pi x}{\lambda}\right) dx$$

$$= \frac{\cancel{2}\pi^{\cancel{2}} A^2 T}{\lambda^{\cancel{2}}} \frac{\cancel{\lambda}}{\cancel{2}}$$

$$= \frac{\pi^2 A^2 T}{\lambda}$$ and since $v = \left(\frac{T}{\mu}\right)^{1/2}$ (μ: mass per unit length)

$$= \frac{\pi^2 A^2 \nu^2 \lambda^{\cancel{2}} \mu}{\cancel{\lambda}}$$ $v^2 \mu = T$, $\nu^2 \lambda^2 \mu = T$

$$= \pi^2 A^2 \mu \lambda \nu^2$$

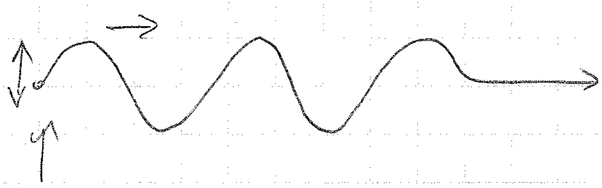
Thus total energy per unit wavelength:

$$E = \underbrace{\pi^2 \mu A^2 \lambda \nu^2 + \pi^2 A^2 \mu \lambda \nu^2}_{\text{equal!}}$$

$$= 2\pi^2 \mu A^2 \lambda \nu^2 \qquad \left(= \frac{1}{2}(\lambda\mu)u_0^2\right)$$

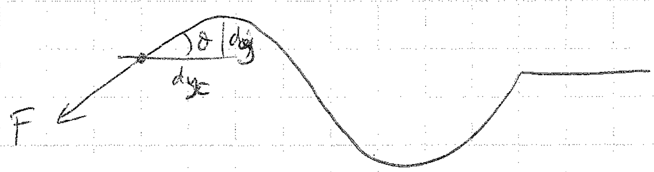
This calculation can be done for any form of wave too, we just follow the same basic procedure with the different function that represent the wave.

Energy transport



Can we calculate the rate of energy input?

Consider the end of the string:



$$y_0 = A \sin 2\pi\nu t$$

$$F_y = -T \sin\theta$$

unlike as before we need to evaluate $\sin\theta$:

F obviously has to be opposite to the tension for the string to oscillate properly.

$$\sin\theta = \frac{dy}{ds}. \quad \text{but } ds = dx\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{1/2}$$

$$\Rightarrow \sin\theta = \frac{\partial y}{\partial s} = \frac{\partial y}{\partial x\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{1/2}} \simeq \frac{\partial y}{\partial x} \cancel{- \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^3 + \ldots}$$

So

$$F_y \approx -T\left(\frac{\partial y}{\partial x}\right)_{x=0}$$

$$\therefore -T\left(\frac{\partial y}{\partial x}\right)_{x=0} = \frac{2\pi\nu A T}{v}\cos 2\pi\nu t$$

using result from earlier

$$\text{Work done} = \int_0^t F.\, dy_0$$

where we've used t to indicate we'll do this over a period

$$= \frac{2\pi\nu A t}{v}\int \cos 2\pi\nu t \; d(A\sin 2\pi\nu t)$$

$$= \frac{(2\pi\nu A)^2 T}{v}\int \cos^2 2\pi\nu t \; dt$$

Since $u_0 = 2\pi\nu A$

$$W = \frac{u_0^2 T}{v} \int \cos^2 2\pi\nu t \, dt$$

set limits $t=0$ to $t = 1/\nu$, ie over 1 cycle, find

$$W_{cycle} = \frac{u_0^2 T}{2v\nu}$$ & since $T = \mu v^2$ $\nu = \frac{v}{\lambda}$

$$W_{cycle} = 2\pi^2\nu^2 A^2 \lambda \mu$$

Which should not be a surprise!

Have just shown that the amount of work put in over 1 cycle = amount of ~~energy~~ in 1 cycle that we previously calculated!

The power is of course energy input divided by time, in this case $\frac{1}{\nu}$

$$\Rightarrow \quad P = \frac{W_{cycle}}{\left(\frac{1}{\nu}\right)} = \frac{u_0^2 T}{2\cancel{\nu} v} \cdot \cancel{\nu} = \frac{u_0^2 T}{2v}$$