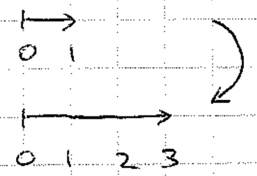
Rotating Vector Representation of SHM

First a little background: Numbers as transformations.

Geometrically we can change objects by stretching, displacing, reflecting and rotating. Lets look at what multiplying by x means on the real number line:

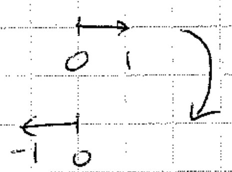
Let's start with $\underline{1}$ & transform this to 3.



this is a stretch

So $\underline{1} \times x$ represent a stretch, if x is +ve & > 1
If x is +ve & < 1 it is a shrink. (Note stretch/shrink = dilation)

What if x is -ve? ie take $x = \underline{-1}$



this is a reflection around 0.

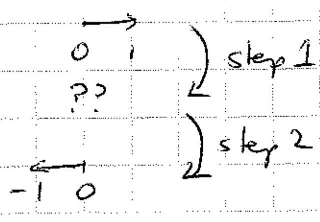
If " x is -ve but not equal to -1 then there will be dilation as well.

So $\underline{1} . x = \underline{-1}$ where $x = \underline{-1}$ gives a reflection.

Two of these reflections returns $\underline{1}$ again. But that is just two negative numbers multiplied together, which you know has to be +ve.

Question: Suppose we apply two transformations (both the same) but we want to get $\underline{-1}$ from $\underline{1}$ - what kind of geometric operation would we do?

i.e.

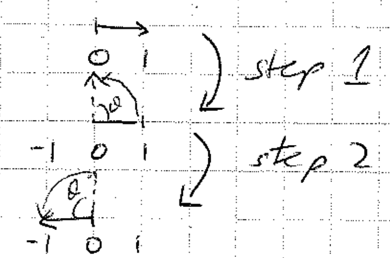


Can't be a dilation – whether done with +ve or -ve numbers end result is +ve.

Can't be reflection – covered above.

Displacements aren't interesting here either (they correspond to a different operation – addition & subtraction).

So the only thing left is? Rotation!



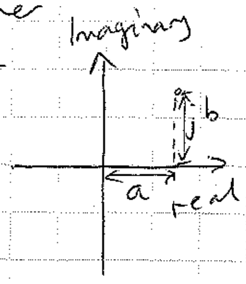
(Could rotate the other way as well.)

But intermediate state does not make sense on the real number line. So we must "imagine" a new number plane that is 2-d rather than 1-d.

Consider the equation $1 \times x \times x = -1$
then $x = \sqrt{-1}$

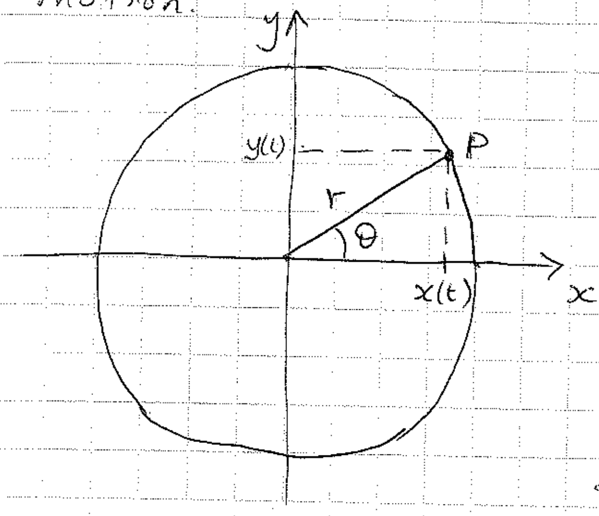
and we call x an imaginary number.
The number $\sqrt{-1}$ is often written as i or j.

Numbers written $a + jb$ are called "complex numbers" and have both real & imaginary parts. Having two components means that they are most naturally described in a 2d plane rather than 1d number line. This plane is often called the "Argand" or complex plane.



Demo: rotating vector video.

SHM can be described as the projection of circular motion.



(See fig 1.4)

For rotational motion θ is changing in time

Set $\theta(t) = \omega t + \alpha$

at $t = 0$ $\theta = \alpha$, so α is the initial angle.

Applying trigonometry to the circle, then

$$x(t) = r\cos\theta(t) = r\cos(\omega t + \alpha)$$
$$\text{and } y(t) = r\sin\theta(t) = r\sin(\omega t + \alpha)$$

But we said in lecture 1 that SHM was described by $x = A\sin(\omega t + \varphi_0)$ how does this relate to the cosine formula for $x(t)$?

Since $\cos\theta = \sin(\theta + \frac{\pi}{2})$ we can equate as follows:

$$A\sin(\omega t + \varphi_0) \equiv r\cos(\omega t + \alpha)$$

Identifying $A = r$ first we then have

$$\sin(\omega t + \varphi_0) = \cos(\omega t + \alpha) = \sin(\omega t + \alpha + \frac{\pi}{2})$$

So we immediately identify $\varphi_0 = \alpha + \frac{\pi}{2}$ although actually there could be an arbitrary number of factors of 2π in here.

Note: Even though the treatment above is 2d we are only extracting 1d information from it. We are "projecting out" 1d information from 2d, so there exist "unphysical" parts in this mathematical description (in this case $y(t)$ has no physical meaning).

Rotating Vectors & Complex Numbers

We can also think of the position P, which has coordinates $(x(t), y(t))$ as being described by a vector from the origin, $\vec{z}$.

If we let the unit vector along x-axis be $\hat{i}$ & along the y-axis be $\hat{j}$ then

$$\vec{z} = x\hat{i} + y\hat{j}$$

We can also describe the position of P in the complex plane as follows:

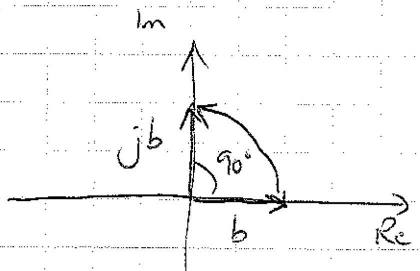
$$z \equiv \text{"}\vec{z}\text{"} = x + jy$$

note: " " are used because you should never equate a vector to a number.

This equivalence works if we think of displacements along the x-axis as implicitly including the unit vector $\hat{i}$, while the imaginary number j signals a displacement along the y-axis.

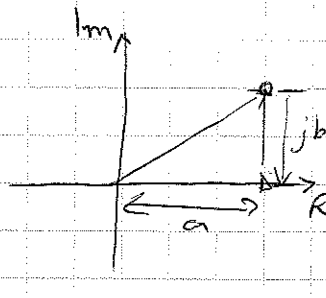
This description, ie using complex numbers, naturally encompasses the idea of rotation for the reasons we discussed earlier, ie

$j \equiv$ perform a rotation of $90°$ to whatever value j multiplies:

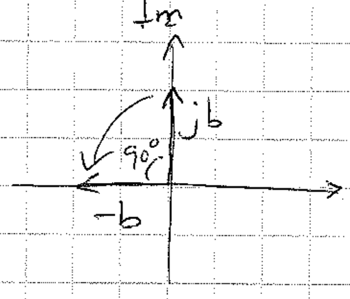


(1) jb = measure b along x-axis & rotate anticlockwise by $90°$

(2) $a + jb$ has both a real (a) & imaginary part (jb)



(3) $j^2 b = j(jb)$ so that rotates jb back on to the real number line.



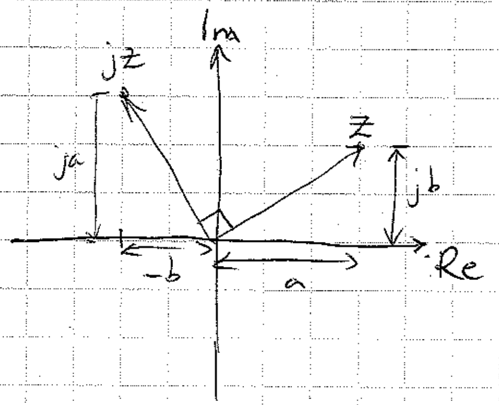
$\Rightarrow \quad j^2 b = -b$

$\Rightarrow \quad j^2 = \sqrt{-1}$ as we showed earlier.

(4) What about jz?

$$jz = j(a+jb)$$
$$= ja + j^2 b$$
$$= ja - b = -b + ja$$

So it is the 90° anticlockwise rotation again:



So given j is just one complex number that rotates things around, are there others that rotate by different angles – yes!! A remarkably beautiful relation allows us to describe which complex number describes rotation by a given angle:

Euler's Formula (1748)

$$\cos\theta + j\sin\theta = e^{j\theta}$$ (see p.13 for proof)

This is an incredibly useful result! It also hides a really cool relationship:

Set $\theta = \pi$ Then

$$-1 + j \times 0 = e^{j\pi}$$

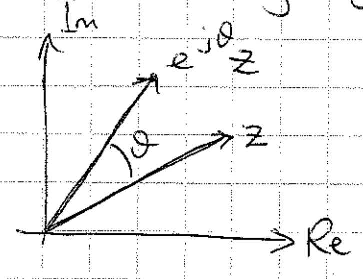
$$\Rightarrow \quad 1 + e^{j\pi} = 0 \qquad \text{(often seen as } 1 + e^{i\pi} = 0\text{)}$$

"Euler's Identity"

$\Rightarrow$ A formula encompassing 5 of the most important numbers in math! Appears on t-shirts quite a lot!

Euler's formula is great because it shows that:

- multiplying a complex number z by $e^{j\theta}$ is equivalent to rotating by an angle θ anticlockwise. [Homework problem]



- connection between rotating vectors in the complex plane & SHM

- integration or differentiation makes $e^{j\theta}$ re-appear:

$$\frac{d}{d\theta} e^{j\theta} = j e^{j\theta}$$

Together, this gives a beautifully clean way to deal with descriptions of SHM.

"Old" way:
$$x = A\cos(\omega t + \varphi_0)$$
$$\dot{x} = \omega A \sin(\omega t + \varphi_0)$$
$$\ddot{x} = -\omega^2 A \cos(\omega t + \varphi_0)$$
$$\Rightarrow \quad \ddot{x} = -\omega^2 x$$

Now:
$$z = A\cos(\omega t + \varphi_0) + jA\sin(\omega t + \varphi_0)$$
$$= A e^{j(\omega t + \varphi_0)}$$
$$\dot{z} = j\omega A e^{j(\omega t + \varphi_0)} = j\omega z$$
$$\ddot{z} = (j\omega)^2 A e^{j(\omega t + \varphi_0)} = -\omega^2 z$$

Then we take the real part for our solution.