Superposition & Dispersion

Let's quickly revisit what we saw last time. For two solution of the wave equation

$$y_1 = A \sin\left[\frac{2\pi}{\lambda_1}(x - vt)\right] \qquad v = \sqrt{\frac{T}{\mu}}$$

$$y_2 = A \sin\left[\frac{2\pi}{\lambda_2}(x - vt)\right]$$

then we know $y = y_1 + y_2$ is also a solution

ie. $$y = A\left\{ \sin\left[\frac{2\pi}{\lambda_1}(x-vt)\right] + \sin\left[\frac{2\pi}{\lambda_2}(x-vt)\right]\right\}$$

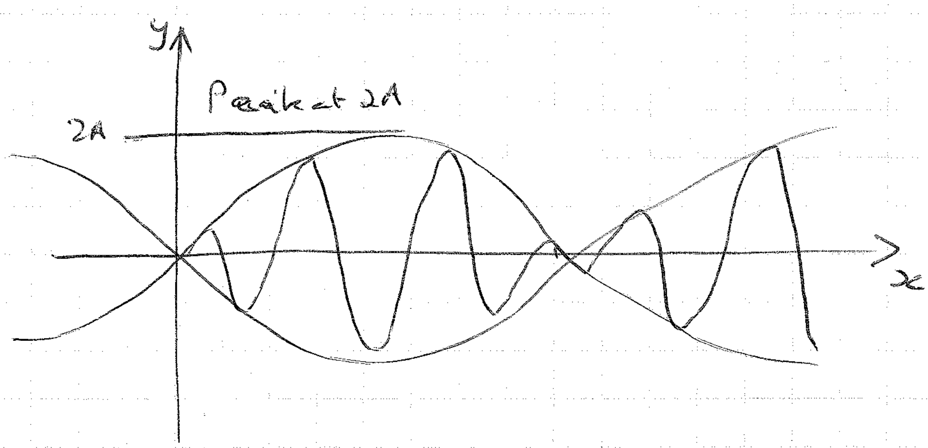
Simplifying by setting $t=0$ and using

$$\sin\theta + \sin\phi = 2\sin\frac{(\theta+\phi)}{2}\cos\frac{(\theta-\phi)}{2}$$

$$y = A\left\{ \sin\left[\frac{2\pi}{\lambda_1}(x-vt)\right] + \sin\left[\frac{2\pi}{\lambda_2}(x-vt)\right]\right\}\Bigg|_{t=0}$$

$$= 2A\sin\left[\pi\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)x\right]\cos\left[\pi\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)x\right]$$

Yields something like this



Note: There are beats in spatial coordinate Not time! ie and $x = x_1 + x_2 = A\cos\omega_1 t + A\cos\omega_2 t$

Wave number:

Let's introduce this quickly - the wavenumber defines the number of wavelengths per unit interval in distance.

$$k = \frac{1}{\lambda} \qquad \text{units } m^{-1}$$

As an example consider visible light $\lambda \sim 500\,nm = 5\times10^{-7}\,m$

$$k = \frac{1}{5\times10^{-7}} = 0.2\times10^{7}\ m^{-1}$$

<u>Note</u> It is actually very common for people to define the angular wavenumber

$$k = \frac{2\pi}{\lambda}$$ just be aware of this for future courses

Using this definition we see that the beat solution becomes

$$y = A\left\{\sin 2\pi k_1 x + \sin 2\pi k_2 x\right\} = 2A\sin(\pi(k_1+k_2)x)\cos(\pi(k_1-k_2)x)$$

Q: What is the peak to peak distance of the modulating function?

A: The modulation is provided by the cos term. Since the peak to peak corresponds to half a wavelength that corresponds to

$\pi(k_1-k_2)D = \pi$ where D is the distance

$$\Rightarrow \quad D = \frac{1}{k_1-k_2} = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}$$

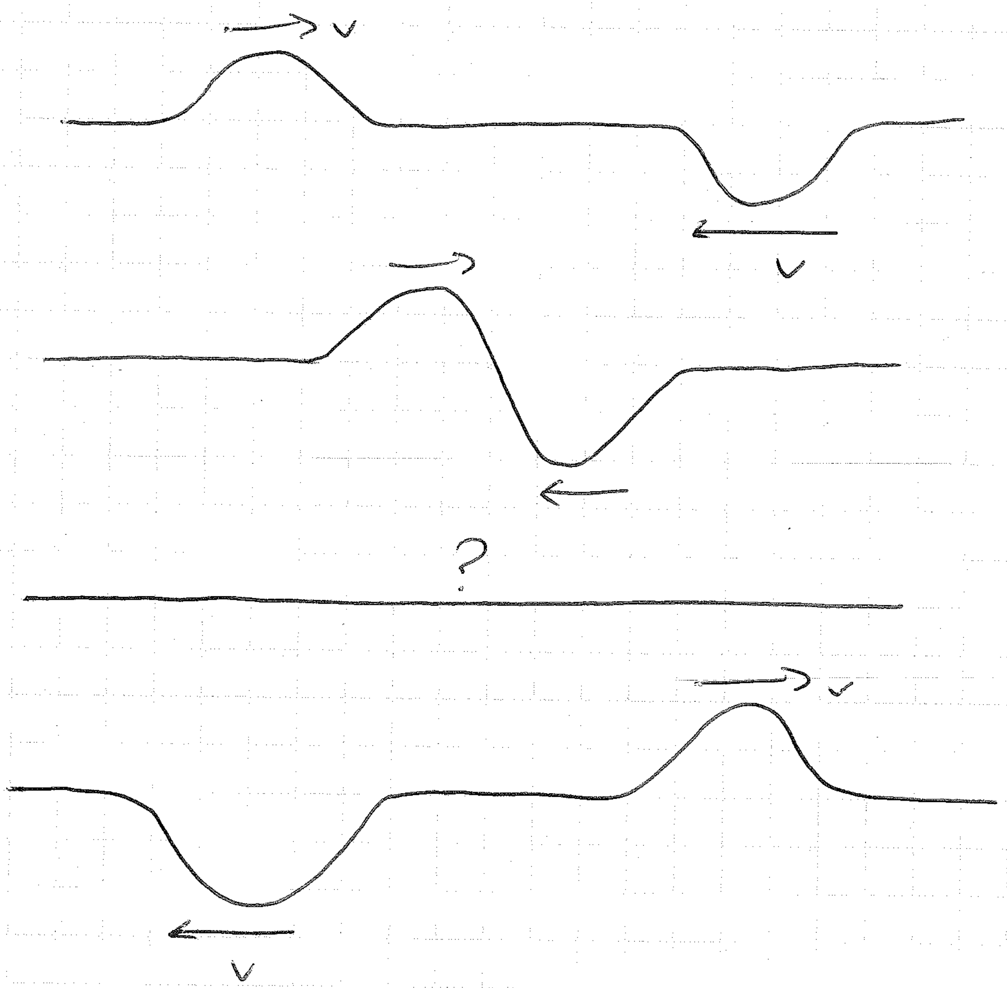
For wavelengths that are close, ie. λ & $\lambda+\Delta\lambda$

Then $D \approx \dfrac{\lambda^2}{\Delta\lambda} = \lambda\left(\dfrac{\lambda}{\Delta\lambda}\right)$ ie approx $\dfrac{\lambda}{\Delta\lambda}$ wavelengths are between peaks.

Wave Pulses – Superposition

We've already discussed how we can set up arbitrary wave pulses using Fourier series.

What happens when we look a superposition of two pulses?



Q: What happens to the energy of the two pulses during the superposition?

A: This is an awesome question!
If you look at the vertical velocity of the string when they perfectly suppose although everything is at $y=0$, the velocities <u>are</u> non-zero!

Consider ~~adding~~ subtracting! $\sin[(x-vt)2\pi] - \sin[(x+vt)2\pi]$
& look at $\frac{\partial}{\partial t}$, the terms add rather than subtract!

Dispersion; Phase & Group Velocities

Thus far we have assumed that the speed of propagation of a wave does not depend on its wavelength (or frequency)

This situation is actually very uncommon. Most media transmit waves of different wavelengths at different speeds.

e.g. ocean waves
radio waves in/interstellar medium
(pulsar "dispersion measure" allows us to calculate distances to pulsars)

For sine waves, ie

$$y(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right] \qquad v = \sqrt{\frac{T}{\mu}}$$

for example

The effect is straightforward
⇒ whole train moves at different speeds.

BUT Pulses created via a Fourier series expansion have different wavelengths!

⇒ Pulses will spread out in time!

So you can think of dispersion as

"That which once was together no longer is"

Let's bring in the math now. Consider two waves with $\lambda_1 \neq \lambda_2$ $v_1 \neq v_2$

Then $y_1 = A \sin\left[\frac{2\pi}{\lambda_1}(x - v_1 t)\right]$

let $\frac{v_1}{\lambda_1} = \nu_1$ Then

$\frac{1}{\lambda_1} = k_1$ $\quad y_1 = A \sin\left[2\pi(k_1 x - \nu_1 t)\right]$

Similarly we set

$$y_2 = A \sin\left[2\pi\left(k_2 x - \nu_2 t\right)\right]$$

Remember: $V_1 = \frac{\nu_1}{k_1} \neq V_2 = \frac{\nu_2}{k_2}$ ie different speeds

Applying our favourite trig formula:

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$y = y_1 + y_2 = A\left[\sin 2\pi(k_1 x_1 - \nu_1 t) + \sin 2\pi(k_2 x_2 - \nu_2 t)\right]$$

$$y = 2A \cos \pi\left[(k_1 - k_2)x - (\nu_1 - \nu_2)t\right]$$
$$\times \sin 2\pi\left[\left(\frac{k_1 + k_2}{2}\right)x - \left(\frac{\nu_1 + \nu_2}{2}\right)t\right] \quad \text{——(I)}$$

Q: What does this look like at $t = 0$?

A: $y = 2A \cos\left[\pi(k_1 - k_2)x\right] \sin\left[\pi(k_1 + k_2)x\right]$

ie just like the beats system.

But you'd expect that because we've not allowed the waves to propagate yet.

Q: What happens as we allow t to evolve?

A: Go back to (I)

Notice how the terms ahead of t, (multiplying) which governs the speed of propagation are different.

ie we have $(\nu_1 - \nu_2)t$ & $(\nu_1 + \nu_2)t$

Notice both waves move in the same direction

We can make the analysis more interesting if we assume the two combined wavelengths are almost the same:

Let $k_1 - k_2 = \Delta k$ $\qquad$ $\nu_1 - \nu_2 = \Delta \nu$

$$\frac{k_1 + k_2}{2} = k \qquad \frac{\nu_1 + \nu_2}{2} = \nu$$

where k & ν are essentially "central" wave numbers & frequencies.

Using these substitutions we get

$$y = 2A \cos \pi (x \Delta k - t \Delta \nu) \sin 2\pi (kx - \nu t)$$
$$= 2A \cos \pi \Delta k \left(x - \frac{\Delta \nu}{\Delta k} t\right) \sin 2\pi k \left(x - \frac{\nu}{k} t\right)$$

Notice the two velocities:

$$\frac{\nu}{k} = v_p = \nu \lambda \qquad \text{"Phase velocity"}$$

Speed at which the crest of the average wave number moves

$$\frac{\Delta \nu}{\Delta k} = v_g \equiv \frac{d\nu}{dk} \qquad \text{"Group velocity"}$$

Velocity at which the modulating envelope moves

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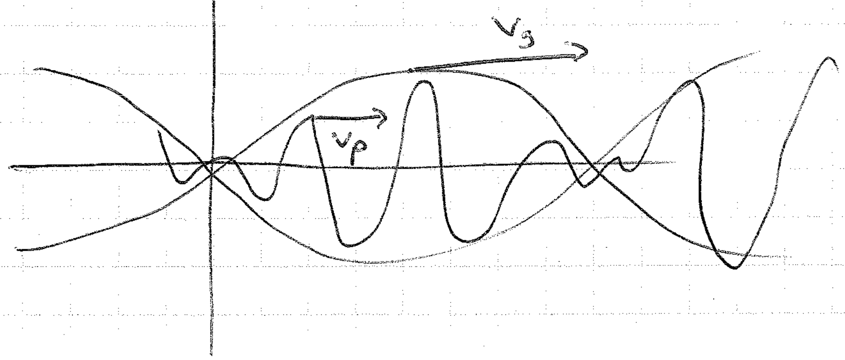

For deep ocean waves "gravity waves" there is a strong dispersion relationship.

$$v_p = v_p(\lambda) = C\lambda^{1/2} = Ck^{-1/2}$$

by definition $v_p = \frac{\nu}{k}$

$$\Rightarrow \quad \frac{\nu}{k} = Ck^{-1/2} \qquad \text{i.e.} \quad \nu = Ck^{1/2}$$

We can now calculate the group velocity:

$$v_g = \frac{d\nu}{dk} = \frac{1}{2}Ck^{-1/2} = \frac{1}{2}v_p$$

Net effect is smaller crests move quickly through ~~the~~ a larger group.