Chapter 7 - Travelling Waves

Thus far we have been concerned with stationary waves

- There is y variation but the extent of the wave in x is fixed

But most waves <u>move</u> or travel, and are unsurprisingly known as either travelling or progressive waves.

Q: Can you give examples of travelling waves?

A: Disturbances on ropes or slinky
Sound
Seismic waves
Ocean waves ...

Ocean waves are actually a good example.

(1) Travel very large distances 100s or even 1000s of km
(2) They are disturbances that propagate through the medium - the medium itself does not move in the direction of propagation
(3) But "something" is still transmitted a long distance via the medium
(4) This is a <u>local effect</u> with a <u>distant cause</u>

Q: What is the connection between the two?
A: Time delay & velocity of propagation

(5) Ocean waves can do damage
$\Rightarrow$ Waves still carry <u>energy</u>.

Although it may seem slightly boring waves on a spring like a slinky ~~are~~ are a good way to study this phenomena.

Demo: Setting up standing waves on a slinky

- Notice (recall) how initial disturbance <u>propagates</u>

As we saw in the demo, the standing wave takes a while to be established. At the ends of the slinky we have a fixed point so the wave reflects back, and ends up interfering with another wave coming down the spring.

Let's examine this mathematically:

Recall we derived the equation of the standing waves to be,

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$$

I've dropped phase here for simplicity

but since we showed $\omega_n = \frac{n\pi}{L}v = \frac{n\pi}{L}\left(\frac{T}{\mu}\right)^{1/2}$

where $\frac{T}{\mu}$ has units of v^2 so we just give $\left(\frac{T}{\mu}\right)^{1/2}$ the symbol v

Using the trig relationship

$$\sin\theta\cos\phi = \frac{1}{2}\left[\sin(\theta+\phi) + \sin(\theta-\phi)\right]$$

we can rewrite

$$\sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) = \frac{1}{2}\left[\sin\left(\frac{n\pi x}{L} - \omega_n t\right) + \sin\left(\frac{n\pi x}{L} + \omega_n t\right)\right]$$

and substituting for ω_n we get

$$y_n(x,t) = \frac{1}{2}A_n \sin\left[\frac{n\pi}{L}(x - vt)\right] + \frac{1}{2}A_n \sin\left[\frac{n\pi}{L}(x+vt)\right]$$

This is actually a really cool result even though it may not look like it to begin with.

Let's examine the first term ie we'll look at

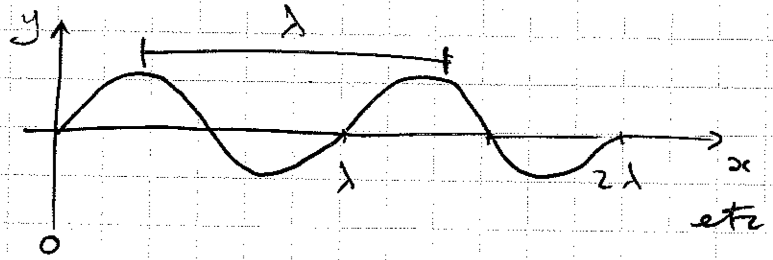
$$y(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

where we've set $A = \frac{1}{2}A_n$, $\lambda = \frac{2L}{n}$

If we look at $t=0$ to begin with what do we have?

$$y(x,0) = A \sin\left[\frac{2\pi x}{\lambda}\right]$$

So as x goes from 0 to λ we cover one entire cycle – λ is obviously the wavelength here.



Q: What happens at a later time?

A: Lets consider the origin initally, ie. $x=0$

As t evolves the vt term will become signifieat
Let's track $y=0$ with time.

Since we know $y=0 \Rightarrow \sin\left[\frac{2\pi}{\lambda}(x-vt)\right]=0$

This zero value must track with $x-vt=0$

$\Rightarrow$ It moves as $x=vt$

$\Rightarrow$ The whole wave is moving with speed v!

We can do the same thing by considering the peak of the wave. At that point the x-derivative will be zero, ie we want to see how $\frac{\partial y}{\partial x}=0$ moves with time.

$$\frac{\partial y}{\partial x} = A.\frac{n\pi}{L}\cos\left[\frac{2\pi}{\lambda}(x-vt)\right] = 0$$

For this result to be true we must have

$$\cos\left[\frac{2\pi}{\lambda}(x - vt)\right] = 0$$

$$\Rightarrow \quad \frac{2\pi}{\lambda}(x - vt) = \frac{\pi}{2} + \cancel{n\pi}$$

drop the n part so we consider only the first peak

$$\Rightarrow \quad x - vt = \frac{\lambda}{4}$$

$$\Rightarrow \quad x = \frac{\lambda}{4} + vt$$

So the peak initially at $\frac{\lambda}{4}$ moves with speed vt - exactly what we just saw.

What about the term

$$\frac{1}{2} A_n \sin\left[\frac{n\pi}{L}(x + vt)\right] \; ?$$

Since the $-vt$ term acted like a phase pushing the wave along in the +ve x-direction..

It should be clear that $+vt$ corresponds to a wave moving in the opposite direction!

But! We started with a standing wave equation - what has happened?

Answer: We've shown that a single standing wave is actually the sum of two travelling waves moving in opposite directions!

Cool!

Progressive waves in one direction

We have just shown that

$$y(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

describes a travelling wave moving in the +ve x direction

By considering derivative of this formula we can see what differential equation governs its evolution.

$$\frac{\partial y}{\partial x} = \frac{2\pi}{\lambda} A \cos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

& $$\frac{\partial^2 y}{\partial x^2} = -\left(\frac{2\pi}{\lambda}\right)^2 A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

While for the time derivatives:

$$\frac{\partial y}{\partial t} = -\frac{2\pi v}{\lambda} A \cos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$$\frac{\partial^2 y}{\partial t^2} = -\left(\frac{2\pi v}{\lambda}\right)^2 A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

There is clearly an obvious similarity between $\frac{\partial^2 y}{\partial x^2}$ & $\frac{\partial^2 y}{\partial t^2}$ and we find

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

or $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$ —— The wave equation again!

This is exactly identical to what we found for the string in chapter 6. This suggests the two are very closely related.

Note: Read 209-211 on longitudinal waves

Superposition of waves

We know that the wave equation is linear, so any sum of two solutions is also a solution.

For $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2}$ we have seen that

$$y(x,t) = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$ is a solution

But this solution works for any wavelength λ! So we could write down two solutions

$$y_1 = A \sin\left[\frac{2\pi}{\lambda_1}(x - vt)\right]$$

$$y_2 = A \sin\left[\frac{2\pi}{\lambda_2}(x - vt)\right]$$

and the sum $y = y_1 + y_2$ is also a solution! ie

$$y = A\left\{ \sin\left[\frac{2\pi}{\lambda_1}(x - vt)\right] + \sin\left[\frac{2\pi}{\lambda_2}(x - vt)\right]\right\}$$

is also a solution

You should recognize that the sum of two sines can give a beat situation:

Since $\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

Taking $t=0$ we find

$$\sin\frac{2\pi x}{\lambda_1} + \sin\frac{2\pi x}{\lambda_2} = 2\underbrace{\sin\left[\pi x\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\right]}_{\text{net frequency given here}}\underbrace{\cos\left[\pi x\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\right]}_{\text{envelope}}$$

You can make arbitrarily complicated waves by adding more & more complicated series of terms. e.g. Fig 7-7 shows wave forms associated with musical instruments.

Wave Pulses

To date we've looked at waves as being a series of crests & troughs that is in some sense infinite.

But! In reality many disturbances are finite in length. We can find solutions to the wave equation that satisfy this idea although it may be necessary to construct them via Fourier analysis.

DEMO: Pulse on slinky

Recall that we showed we could write the variation of a time dependent displacement as

$$y(t) = \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \delta_n)$$

$$= \sum_{n=1}^{\infty} C_n \cos(n\omega_1 t - \delta_n) \qquad \text{since } \omega_n = n\omega_1$$

This allows us to oscillate the end of a system was a completely arbitrary wave form.