Fourier Series

We're going to touch on this very briefly, but in all honesty it deserves an entire course!

At its heart, Fourier analysis is about taking something complex, ie a waveform, and decomposing it into simpler pieces. Approximations may be involved, but the goal of the method is to make analysis easier - even if it might not seem that way when you first see it!

The applications of Fourier analysis are vast:

- Math - solutions of PDE's
- Engineering - signal processing
- Image analysis & compression (JPEG)
- Acoustics, heat flow
- even mathematical finance

Here we'll just touch on the most basic concept, that of <u>Fourier series</u>. We'll show how we can mimic a function with a series of sin functions.

Recall for free vibrations of the string we would have solution

$$y_n(x,t) = \underbrace{A_n \sin\left(\frac{n\pi x}{L}\right)}_{\text{amplitude envelope}} \underbrace{\cos(\omega_n t - \delta_n)}_{\text{time evolution including phase } \delta_n}$$

where $\omega_n = \dfrac{n\pi v}{L} = \dfrac{n\pi}{L}\left(\dfrac{T}{\mu}\right)^{1/2}$

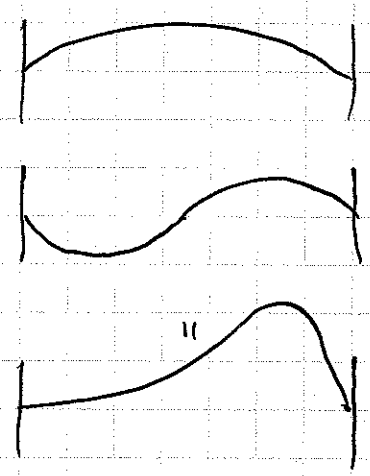
The wave equation itself is linear in y:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

i.e. there are no terms other than y^1

This means that any linear combination of y_n is also a solution

e.g. $y = y_1 + y_2$



So this actually means that any linear combination of all the y_n is also a solution i.e.

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$$

Is actually the most general solution possible. Phases & amplitudes will be set by initial conditions, although as we'll see the continuous nature of the system really means we need to specify quite a lot of information for an arbitrary configuration.

Let's simplify first:

Imagine what we have at a specific moment in time t_0 - imagine a flash shot of the system.

With time frozen we only have a spatial dependence

$$y(x,t_0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

really $B_n(t_0) = A_n \cos(\omega_n t_0 - \delta_n)$

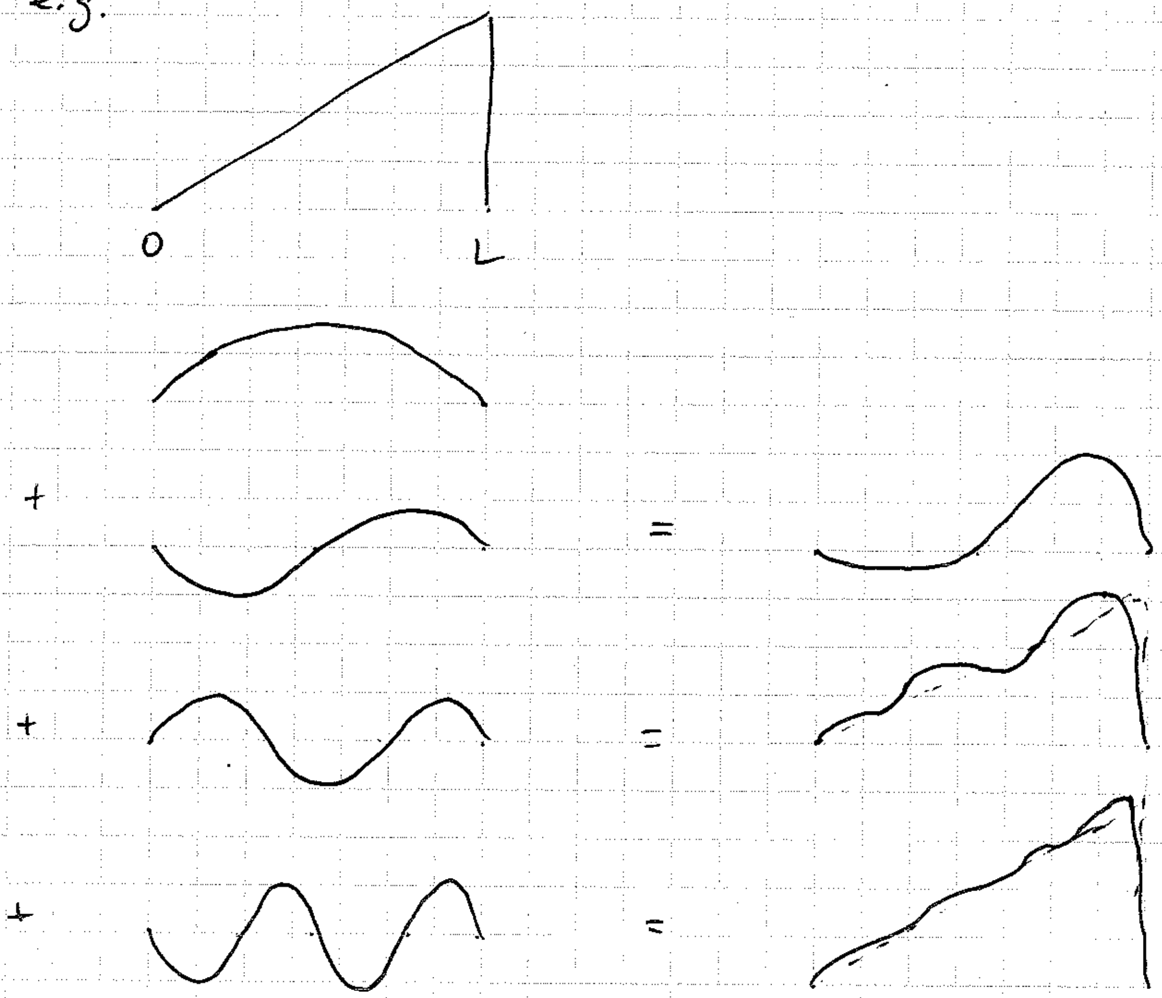
But these are constants at t_0

This is a very neat result:

Any profile of the string $y(x)$ between $x=0$ & $x=L$ (subject to $x(0)=0$, $x(L)=0$) can be represented as an infinite series of sine functions as per

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

e.g.



Q: Can you see a way to apply this idea in the x-direction?

A: Now "freeze" x but look at variation wrt t:

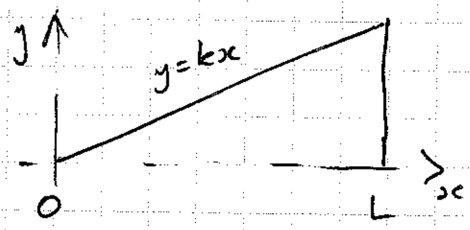
$$y(x_0, t) = \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \delta_n)$$

$$\underbrace{C_n \equiv C_n(x_0) = A_n \sin\left(\frac{n\pi x_0}{L}\right)}_{\text{constants}}$$

This represents Fourier analysis in time while our first example was Fourier analysis in space.

Returning to the spatial example - the obvious question is how do we calculate the constants?

Lets use the linear slope example:



and try to express as

$$(kx =) y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad (\mathrm{II})$$

So what we need is an expression for B_n for any value of n

Mathematically we want to derive an expression for B_n based upon

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

This looks difficult! We have an infinite series on the right. The idea we use is similar to the concept of "projecting out" values on to the x-axis. Except here we have an infinite number of variables - gulp! :-)

Thus we want to apply some kind of mathematical approach to both sides that will just give us a B_n term on the RHS and something else on the LHS.

We shall multiply by $\sin\left(\frac{m\pi x}{L}\right)$ & then integrate from 0 to L:

$$\int_0^L y(x) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

Need to simplify RHS.

Use $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$
$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$$\therefore \cos(\theta-\phi) - \cos(\theta+\phi) = 2\sin\theta\sin\phi$$

Using this to substitute we get

$$RHS = \frac{1}{2}\sum_{n=1}^{\infty} B_n \int_0^L \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right] dx$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} B_n \left[\frac{L}{\pi(n-m)}\sin\left(\frac{(n-m)\pi x}{L}\right) - \frac{L}{\pi(n+m)}\sin\left(\frac{(n+m)\pi x}{L}\right)\right]_0^L$$

Restrict m to be on $[1,\infty)$ we might be tempted to conclude the bracketed term is always zero! Consider the second term:

$$\sin\left(\frac{(n+m)\pi L}{L}\right) = \sin\left((n+m)\pi\right) = 0$$

For the first term you might also assume

$$\sin\left(\frac{(n-m)\pi L}{L}\right) = \sin\left((n-m)\pi\right) = 0$$

BUT!! what about $n-m=0$?
Then we appear to have $\frac{0}{0}$ & must calculate for first principles again. <u>But</u> there's only one term that matters now!:

$$\therefore RHS = B_m \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx \quad (\text{ie } n=m)$$

(note $\frac{1}{2}$ disappeared because we no longer need cos substitution)

$$= B_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right) dx = B_m \int_0^L \frac{1}{2}\left(1 - \cos\left(\frac{2m\pi x}{L}\right)\right) dx$$

$$= \frac{1}{2} B_m \left[x - \cancel{\frac{L}{2m\pi} \sin\left(\frac{2m\pi x}{L}\right)} \right]_0^L$$

$$= \frac{1}{2} B_m L$$

And now equating to the LHS side we find:

$$\int_0^L y(x) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2} B_m L$$

$$\Rightarrow \quad B_m = \frac{2}{L} \int_0^L y(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

or equivalently

$$B_n = \frac{2}{L} \int_0^L y(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Example. We set $y = kx$

$$\therefore \quad B_n = \frac{2}{L} \int_0^L kx \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2k}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx$$

Use integration by parts "$\int u\,dv = [uv] - \int v\,du$"

$$= \frac{2k}{L} \left\{ \left[\frac{-Lx}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \right]_0^L + \frac{L}{n\pi} \int_0^L \cos\left(\frac{n\pi x}{L}\right) dx \right\}$$

$$= \frac{2k}{n\pi} \left\{ - \left[x \cos\left(\frac{n\pi x}{L}\right) \right]_0^L + \underbrace{\frac{L}{n\pi} \left[\sin\left(\frac{n\pi x}{L}\right) \right]_0^L}_{0} \right\}$$

$$\Rightarrow B_n = \frac{-2kL\cos(n\pi)}{n\pi}$$

For n even $\cos(n\pi) = 1$

n odd $\cos(n\pi) = -1$

So $\cos(n\pi) = (-1)^n$

$$\therefore \quad B_n = (-1)^{n+1} \frac{2kL}{n\pi}$$

Very cool – now have simple expression for all the B_n!

$$\Rightarrow \quad y(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2kL}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

and writing out the first few terms:

$$y(x) = \frac{2kL}{n\pi} \left\{ \sin\left(\frac{\pi x}{L}\right) - \frac{1}{2} \sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3} \sin\left(\frac{3\pi x}{L}\right) - \dots \right\}$$

Thus you can now express a function via an infinite sum of sinusoids!