Forced Vibrations of a String

Thus far we've looked at free vibrations, with both ends fixed.

But what happens when we drive the system? + How do we actually do that?

< DEMO >

We can actually set things up by keeping $x = L$ fixed, but letting $x = 0$ move by some angular frequency & amplitude.
For the sake of setting things up, we assume the driven end is given by

$$y(0,t) = B \cos \omega t$$

while at the other end

$$y(L,t) = 0$$

Note: Nothing has changed in terms of the general equation of motion we can continue to use

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Because the amplitude is non-zero at $x=0$ we need to have the x-part of the solution ~~to~~ be non-zero at $x=0$.
So we let $f(x) = A \sin(kx + \alpha)$

We know from the free example that K became $\frac{\omega}{v}$ but just checking again: 16.2

$$\frac{\partial^2 y}{\partial x^2} = -K^2 A \sin(Kx+\alpha) \cos\omega t$$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(Kx+\alpha) \cos\omega t$$

$$\therefore \quad -K^2 A \sin(Kx+\alpha)\cos\omega t = -\frac{\omega^2}{v^2} A \cancel{\sin(Kx+\alpha)\cos\omega t}$$

$$\Rightarrow \quad -K^2 A = -\frac{\omega^2}{v^2} A$$

So no surprise we get $K = \frac{\omega}{v}$ again.

Hence the spatial part of the solution is

$$f(x) = A \sin\left(\frac{\omega x}{v} + \alpha\right)$$

In terms of constraining this function we know that at $x = L$ $y(L,t) = 0$

$\Rightarrow$ amplitude is zero at $x = L$

$$\therefore \quad \sin\left(\frac{\omega L}{v} + \alpha\right) = 0$$

Thus just as previously:

$$\frac{\omega L}{v} + \alpha = p\pi \qquad \text{where } p \text{ is an integer}$$

While at $x = 0$ we have

$$A \sin\alpha = B$$

and using $\alpha = p\pi - \frac{\omega L}{v}$ we get

$$A = \frac{B}{\sin\left(p\pi - \frac{\omega L}{v}\right)}$$

Q: When does the amplitude become large?
A: When $p\pi - \frac{\omega L}{v} = 0$ (ignoring $n\pi$ here)

But $p\pi = \frac{\omega L}{v}$ so $\omega = \frac{p\pi v}{L}$ is exactly what we found for the free case.

$\Rightarrow$ Whenever the driving frequency is close to the natural frequency the amplitude increases (divergent exactly at natural frequencies).

In practice of course damping will prevent the amplitudes becoming unrealistically large.

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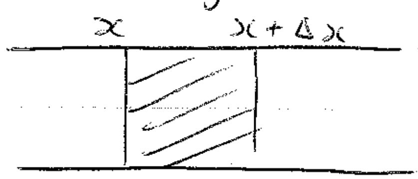
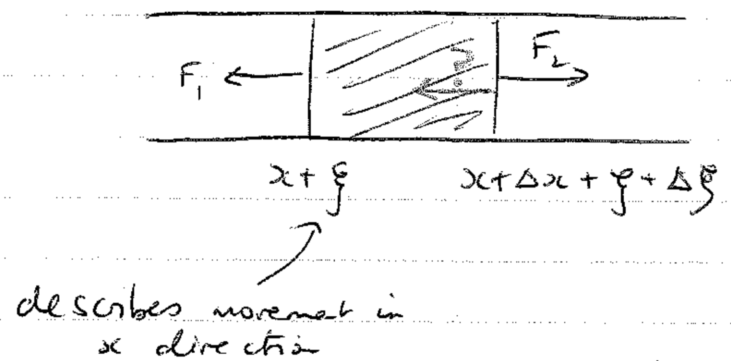
What about longitudinal vibrations?

Vibrations along the length of a rod can also be described using a similar approach.



Piece of a massive rod



Piece of material of fixed mass is displaced via displacement field ξ

The forces F_1 & F_2 actually depend upon fractional change in interatomic separations at x & $x + \Delta x$.

How to attack the problem? Recall we defined the Young's Modulus via

$$\frac{dF/A}{dl/l_0} = -Y$$

where dF defines the force exerted on another object by a wire. (for example)

But by definition

$$\frac{\text{stress}}{\text{strain}} = Y$$

So if we can relate the strain to the changes in position then we can get the stresses via

$$\text{stress} = Y \times \text{strain}$$

For the moved piece of rod the length has increased by $\Delta \xi$

$$\therefore \quad \text{Strain (average)} = \frac{\Delta \xi}{\Delta x}$$

and so $\text{stress} = Y \dfrac{\Delta \xi}{\Delta x}$ at x

$$\therefore \quad \text{Stress} = Y \frac{\partial \xi}{\partial x}$$ (partial because it could vary with t)

What about at $x + \Delta x$?
Since the displacement field ξ is changing to $\xi + \Delta \xi$ its going to be different.

Given the stress field we can say for small changes in x ie Δx that the change in the stress is

$$\frac{\partial}{\partial x}(\text{stress}) \cdot \Delta x$$

$\therefore$ stress at $(x+\Delta x)$ = (stress at x) + $\frac{\partial}{\partial x}(\text{stress } x) \cdot \Delta x$

Putting the cross-section back in (recall stress = $\frac{F}{\text{area}}$)
let α = cross section

$$\Rightarrow \quad F_1 = \alpha Y \frac{\partial \xi}{\partial x}$$

& for F_2:

$$F_2 = \alpha Y \frac{\partial \xi}{\partial x} + \alpha Y \frac{\partial^2 \xi}{\partial x^2} \Delta x$$

Thus

$$F_2 - F_1 = \alpha Y \frac{\partial^2 \xi}{\partial x^2} \Delta x$$

That's really the most difficult part as now we just need the mass of the material.

Given a density ρ, crosssection α & length Δx then clearly $m = \rho \alpha \Delta x$ hence the equation of motion must be

$$F_2 - F_1 = m \frac{\partial^2 \xi}{\partial t^2} = \alpha Y \frac{\partial^2 \xi}{\partial x^2} \Delta x$$

$$\Rightarrow \quad \frac{\partial^2 \xi}{\partial x^2} = \frac{\rho}{Y} \frac{\partial^2 \xi}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 \xi}{\partial t^2}$$

where $v = \left(\frac{Y}{\rho}\right)^{1/2}$

This is clearly really similar to what we had before!
So we consider solutions of the form:

$$\xi(x,t) = f(x)\cos\omega t$$

But what about the end points?
Clamped at both ends doesn't make much sense if you need to strike an end to start the oscillation.
$\Rightarrow$ Could have one end free or both.

Consider one end free for now. Let $x=0$ be fixed & $x=L$ can move.

We know from previous experience with the wave equation we expect solutions of the form

$$f(x) = A\sin\left(\frac{\omega x}{v}\right)$$

This works well for $x=0$ fixed.

What does the free end imply physically?

No other material pulls on the rod at its free end $\Rightarrow$ stress must go to zero there. since the force does.

$$\therefore \quad F = \alpha Y \frac{\partial \xi}{\partial x} = 0$$

Taking $\xi = A\sin\left(\frac{\omega x}{v}\right)\cos\omega t$ & differentiating we find at $x=L$

$$\cos\left(\frac{\omega L}{v}\right) = 0$$

$$\Rightarrow \quad \frac{\omega L}{v} = n\pi - \frac{\pi}{2} = \pi\left(n - \frac{1}{2}\right)$$

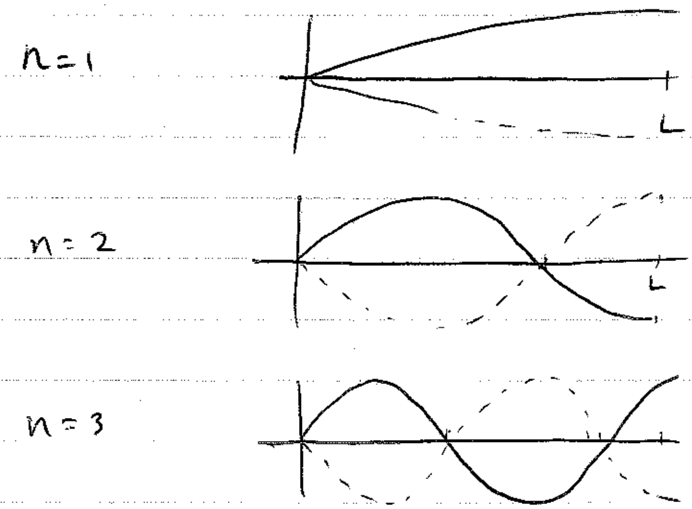
$$\therefore \quad \omega = \frac{\pi v}{L}\left(n - \frac{1}{2}\right)$$

Implying the natural frequencies are

$$\nu_n = \frac{(n-\frac{1}{2})v}{2L} = \frac{(n-\frac{1}{2})}{2L}\left(\frac{Y}{\rho}\right)^{1/2} \Rightarrow \nu_1 = \frac{v}{4L} \; !$$

This is actually different to the string case we analysed due to the free end.

Rather than $\frac{1}{2}$ wavelengths we now define things in terms of $\frac{1}{4}$ wavelengths e.g.

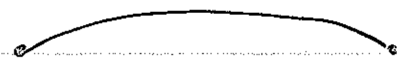


But remember these are longitudinal!!

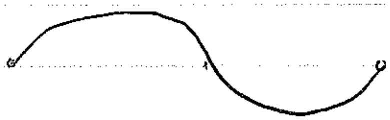
What kind of frequencies could we expect?
For a 1m rod of aluminium

$$Y = 6 \times 10^{10} \ kg\,m^{-1}\,sec^{-2}$$
$$\rho = 2.7 \times 10^{3} \ kg\,m^{-3}$$
$$\Rightarrow \quad \nu_1 \simeq 1200\,Hz \; !$$

We've thus seen two distinct types of modes



$n = 1$



$n = 2$

$n = 3$

In this case $\lambda_n = \frac{2L}{n}$ & $\frac{\omega}{v} = \frac{n\pi}{L} = \frac{2\pi}{\lambda_n}$

and we also saw the 1/4 wave situation that occurred with the free end.

We thus come to two important conclusions:

(1) The boundary conditions play a decisive role in determining the behaviour of the normal modes

(2) Since the equation of motion is linear (ie only terms in x if for longitudinal or y for transverse) then all of the normal modes can co-exist.

Practical note:

The number of modes cannot actually be ∞ as the number of particles in the system is not actually ∞. The proportionality of frequency ν_n to n is a somewhat special case for 1-d that can breakdown in 2 & 3-dimensions.