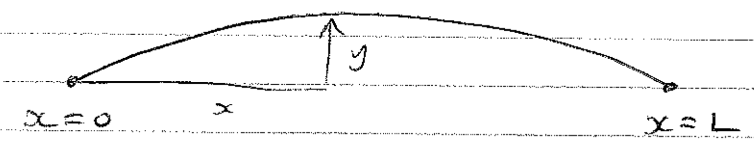
Continuous Systems

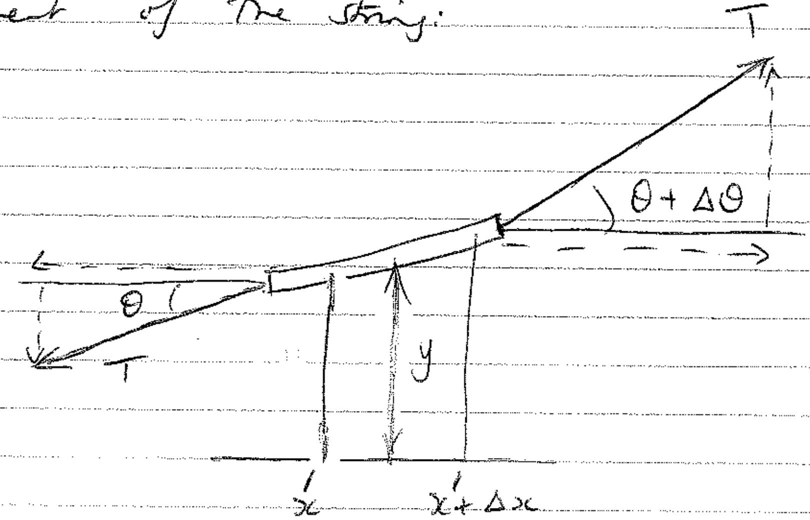
Having considered the problem of N masses, it is now natural to consider an infinite number of masses - a continuous system.

Key assumption: We have a string under tension T



each string element of length Δx has position x, y. The y positions are functions of t.

Consider the change in angle around a small element of the string:



We can now evaluate the forces:

$$F_y = T\sin(\theta+\Delta\theta) - T\sin\theta$$

$$F_x = T\cos(\theta+\Delta\theta) - T\cos\theta$$

Assume y displacements are small $\Rightarrow$ both θ & $\Delta\theta$ are small.

Then we set $F_x \simeq 0$ because $\cos\theta+\Delta\theta \simeq \cos\theta + \underset{\text{term}}{(\Delta\theta)^2}$

For $\sin(\theta + \Delta\theta) \simeq \theta + \Delta\theta$ for small angles (+ $\sin\theta \simeq \theta$)

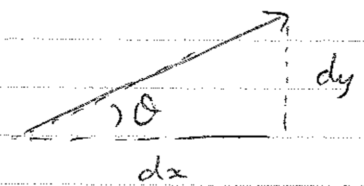
$\therefore \quad F_y \approx T.(\theta + \Delta\theta) - T.\theta = T.\Delta\theta$

Mass of string element = mass per unit length × length
= $\mu \Delta x$

$\Rightarrow \quad \mu \Delta x . a_y = T\Delta\theta$ where a_y = y acceleration

So we have a formula containing $\Delta\theta$ & Δx

At a fixed time t, the angle θ is described as follows:



So $\tan\theta = \dfrac{\partial y}{\partial x}$ (partials because things will change with time)

We can then use this to tell us what will happen for small changes in θ & x (ie $\Delta\theta$ & Δx)

Since $\dfrac{\partial}{\partial x}\tan\theta = \dfrac{\partial^2 y}{\partial x^2}$

$\Rightarrow \quad \dfrac{\partial\theta}{\partial x}\sec^2\theta = \dfrac{\partial^2 y}{\partial x^2}$

But for small changes $\dfrac{\partial\theta}{\partial x} \simeq \dfrac{\Delta\theta}{\Delta x}$

$\therefore \quad \dfrac{\Delta\theta}{\Delta x}\sec^2\theta = \dfrac{\partial^2 y}{\partial x^2}$

and lastly if we again assume small angles $\sec^2\theta \sim 1$

$\Rightarrow \quad \dfrac{\Delta\theta}{\Delta x} = \dfrac{\partial^2 y}{\partial x^2} \Rightarrow \Delta\theta = \dfrac{\partial^2 y}{\partial x^2}\Delta x$

Now substitute this result into the earlier

$$\mu \Delta x \, a_y = T \Delta \theta$$

$$\Rightarrow \quad \mu \cancel{\Delta x} \, a_y = T \frac{\partial^2 y}{\partial x^2} \cancel{\Delta x}$$

$$\Rightarrow \quad \mu a_y = T \frac{\partial^2 y}{\partial x^2}$$

and since $a_y = \frac{\partial^2 y}{\partial t^2}$

$$\Rightarrow \quad \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

Think about units for a second:

$$\frac{\mu}{T} = \frac{M L^{-1}}{M \; L T^{-2}} = L^{-2} T^2 = \frac{1}{\text{velocity}^2}$$

This velocity will actually be associated with waves moving along the string.

For now we write $v = \sqrt{\frac{T}{\mu}}$

$$\Rightarrow \quad \boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}}$$

— The "wave equation" very important in physics!

Let's look for stationary (ie "non-moving") solutions.

In this sense it means the behavior at a given x value is fixed by $\cos \omega t$ behavior.

This implies are very specific form for the trial solution:

$$y(x,t) = f(x) \underbrace{\cos \omega t}_{\text{time dependence}}$$

"Amplitude envelope"

Evaluate derivatives in wave equation:

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x) \cos \omega t$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 f}{dx^2} \cos \omega t$$

Substituting back into wave equation we get:

$$\frac{d^2 f}{dx^2} \cancel{\cos \omega t} = \frac{1}{v^2} \cdot (-\omega^2) f(x) \cancel{\cos \omega t}$$

$$\Rightarrow \quad \frac{d^2 f}{dx^2} = -\frac{\omega^2}{v^2} f(x)$$

ie $$\frac{d^2 f}{dx^2} + \frac{\omega^2}{v^2} f = 0$$

Have seen this before!! Have sine/cosine solutions in x.

Since we know at $x=0$ the system is constrained $f(0)=0$ lets take the sine solution:

let $$f(x) = A \sin\left(\frac{\omega}{v} x\right)$$

Now we also want the right hand end point to be fixed, ie $y(L)=0$

$$\Rightarrow f(L) = 0$$

$$\Rightarrow \quad A \sin\left(\frac{\omega L}{v}\right) = 0$$

So $$\frac{\omega L}{v} = n\pi \quad \text{ie} \quad \omega = \frac{n\pi v}{L}$$

Now we can construct the complete solution.

$$y(x,t) = f(x)\cos\omega t$$

$$= \underbrace{A\sin\left(\frac{\omega x}{v}\right)}_{\text{spatial}}\ \underbrace{\cos\omega t}_{\text{temporal}}$$

Including the n-dependence we write

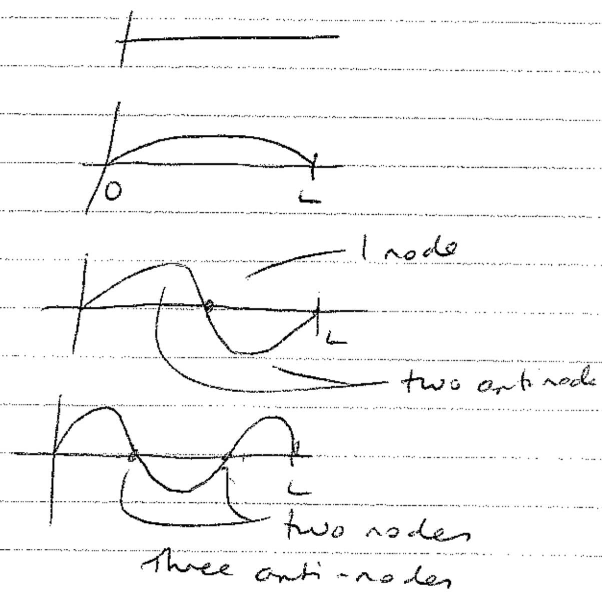
$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v t}{L}\right)$$

The n^{th} mode

Notice now we have an infinite number of modes (at least in theory!)

Visualizing different n:

n	spatial part	Graph
0	0	
1	$A\sin\left(\frac{\pi x}{L}\right)$	
2	$A\sin\left(\frac{2\pi x}{L}\right)$	
3	$A\sin\left(\frac{3\pi x}{L}\right)$	



These shapes are thus setting wavelengths:

$n=1 \qquad \lambda = 2L$

$n=2 \qquad \lambda = L$

$n=3 \qquad \lambda_3 = 2L/3$

$\vdots$

$n \qquad \lambda_n = 2L/n$

What is the temporal evolution doing?

$\cos(\omega t)$ term means that it just oscillates back & forth with the shape always being given by the spatial part.

However the frequency component does show dependence upon n because

$$\omega_n = \frac{n\pi v}{L}$$

In relation to the frequencies,

$$f = \frac{\omega}{2\pi} = \frac{n\cancel{\pi} v}{L}\frac{1}{2\cancel{\pi}} = \frac{nv}{2L}$$

This n dependence means different modes have different frequencies.

The lowest frequency describes the "fundamental" mode

$f_1 = \frac{v}{2L}$ gives the fundamental frequency

$f_n = n f_1$ are overtones (multiples of the fundamental) and are often called harmonics.

What determines the frequency you hear?

$$f_n \equiv \nu_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \Rightarrow$$

Higher n $\Rightarrow$ higher freq.
Higher tension $\Rightarrow$ " "
Heavier string $\Rightarrow$ lower freq
Shorter L $\Rightarrow$ higher freq.

Physics works! ☺