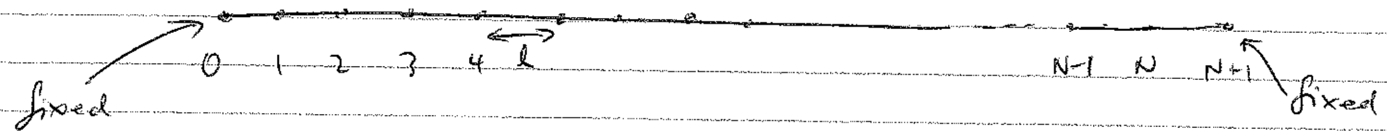
N Coupled Oscillators

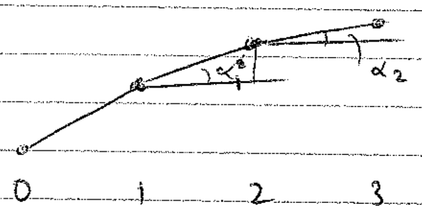
We now switch from longitudinal to transverse oscillation.

Suppose initially a string of particles of mass m is set-up & there are 0 to $N+1$ of them



Two particles at end points are fixed.

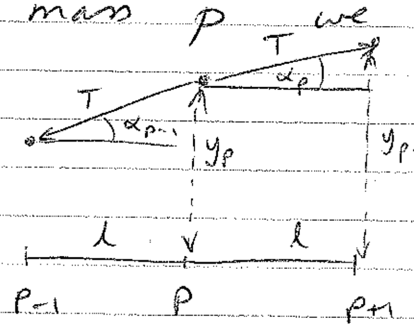
Suppose tension in string is initially T, and then we displace the string:



Each angle describes the offset in position from one particle to the next

Note: increase in length is small provided we consider small displacements - so we ignore it.

At mass p we have



Net x forces:

$$F_p^x = -T\cos\alpha_{p-1} + T\cos\alpha_p$$

Ignore!! provided α is small. These terms in α_{p-1}^2 & α_p^2 will be small too (consider $\cos x \simeq 1 - \frac{x^2}{2}$)

Net y force:

"pulls down" in our diagram

"pulls up" in our diagram

$$F_p^y = -T\sin\alpha_{p-1} + T\sin\alpha_p$$

To evaluate the angles, use simple trig; but let the distance between masses be given by l, so we again ignore the stretch.

Then $$\sin\alpha_{p-1} = \frac{y_p - y_{p-1}}{l}$$

$$\sin\alpha_p = \frac{y_{p+1} - y_p}{l}$$

Thus
$$F_p = T\left[-\left(\frac{y_p - y_{p-1}}{l}\right) + \left(\frac{y_{p+1} - y_p}{l}\right)\right]$$

$$\Rightarrow \quad m\ddot{y}_p = \frac{T}{l}\left\{-2y_p + y_{p+1} + y_{p-1}\right\}$$

$$\Rightarrow \quad \ddot{y}_p = \frac{T}{ml}\left(-2y_p + y_{p+1} + y_{p-1}\right) = \omega_0^2\left(-2y_p + y_{p+1} + y_{p-1}\right)$$

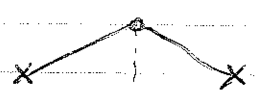
where $\omega_0^2 = \frac{T}{ml}$

So we get

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 y_{p+1} - \omega_0^2 y_{p-1} = 0$$

BUT every particle has one of these equations!
$\Rightarrow$ We have a system of $\underline{\underline{N}}$ equations.

To develop intuition, consider $N=1$



In this case y_{p+1} & y_{p-1} terms are both zero

$$\Rightarrow \quad \ddot{y}_p + 2\omega_0^2 y_p = 0$$

Q: What is angular frequency here?

A: $\omega = \sqrt{2}\, \omega_0 = \left(\dfrac{2T}{ml}\right)^{1/2}$

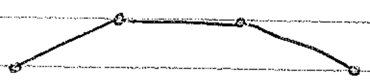
For $N=2$

$$\ddot{y}_1 + 2\omega_0^2 y_1 - \omega_0^2 y_2 = 0 \qquad \text{as } y_0 = 0$$

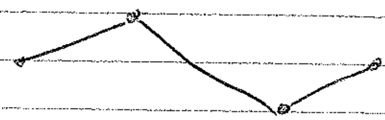
$$\ddot{y}_2 + 2\omega_0^2 y_2 - \omega_0^2 y_1 = 0 \qquad \text{as } y_3 = 0$$

Similar to coupled pendulum equation, although $\omega_0^2 + \omega_c^2$ is now $2\omega_0^2$ – recall normal mode frequencies were $\omega_0^2 + 2\omega_c^2$ & ω_0^2.
From this we can then guess that the angular frequencies will be ω_0^* & $\sqrt{3}\,\omega_0$

Q: What form will the normal modes take in this situation?



Lower



upper

$\omega_0 = \sqrt{\dfrac{T}{ml}}$ $\qquad\qquad$ $\omega_0 \sqrt{3}$

As in coupled pendulum the full solution is a combination of these modes.

But we really want to apply this to the case of N masses $\Rightarrow$ N equations.

We'll look for normal mode solutions of the form

$$y_p = A_p \cos \omega t \qquad p = 1, 2, \ldots, N$$

A_p – amplitude of p^{th} particle

y_p – y position of p^{th} particle

Note: assuming trial solution of this form also ensures that all these particles must start at rest at $t=0$ (due to $\dot{y}_p = -\omega A_p \sin\omega t$).

The equation we derived earlier for the p^{th} particle was

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 y_{p+1} - \omega_0^2 y_{p-1} = 0$$

Substitute the trial solution for all ~~Ap~~ positions on the string:

$$-\omega^2 A_p \cos\omega t + 2\omega_0^2 A_p \cos\omega t - \omega_0^2 A_{p+1} \cos\omega t - \omega_0^2 A_{p-1} \cos\omega t = 0$$

$$\Rightarrow \quad -\omega^2 A_p + 2\omega_0^2 A_p - \omega_0^2 A_{p+1} - \omega_0^2 A_{p-1} = 0$$

$$\Rightarrow \quad (-\omega^2 + 2\omega_0^2) A_p - \omega_0^2 (A_{p+1} + A_{p-1}) = 0$$

AND there are N of these equations!

Remember we set the end points as fixed so $A_0 = A_{N+1} = 0$

We can rearrange as follows:

$$\frac{A_{p+1} + A_{p-1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

Once we have a value of ω^2 the RHS's value is set

$\Rightarrow$ LHS must be the same for all p!

OK this is hard to solve! But let's try the following:

Let $\quad A_p = C \sin(p\theta)$

where θ is an as yet unspecified angle.

Then $A_{p+1} = C \sin[(p+1)\theta]$

$A_{p-1} = C \sin[(p-1)\theta]$

$$\Rightarrow A_{p+1} + A_{p-1} = C \left\{ \sin[(p+1)\theta] + \sin[(p-1)\theta] \right\}$$

$$= 2C \sin(p\theta) \cos\theta$$

work through

$\sin\alpha + \sin\beta = 2 \sin\frac{\alpha+\beta}{2} \cos\frac{\alpha-\beta}{2}$

$$= 2A_p \cos\theta$$

Hence

$$\frac{A_{p+1} + A_{p-1}}{A_p} = \underbrace{2\cos\theta}_{\text{constant independent of } p!} \left(= \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} \right)$$

from earlier

But how do we set θ?

Recall $A_0 = 0$ So $A_0 = C \sin 0.\theta = 0$ works.

$A_{N+1} = 0$ So $\underbrace{A_{N+1} = C \sin(N+1)\theta = 0}$

immediately sets a requirement on θ!!

Must have

$$(N+1)\theta = n\pi$$

$$\Rightarrow \theta = \frac{n\pi}{(N+1)}$$

Therefore our substitution is

$$A_p = C \sin\left(\frac{pn\pi}{(N+1)}\right)$$

The allowed angular frequencies of the normal modes are now set by

$$2\cos\theta = 2\cos\left(\frac{n\pi}{N+1}\right) = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

$$\Rightarrow \quad \omega^2 = 2\omega_0^2\left[1 - \cos\left(\frac{n\pi}{N+1}\right)\right]$$

$$= 4\omega_0^2 \sin^2\left[\frac{n\pi}{2(N+1)}\right]$$

using $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$

$$\Rightarrow \quad \omega = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right]$$

allowed frequency of normal mode

This seems to have a lot of solution i.e. $n = 1, 2, 3, 4$

Think about limits here:

N is set by system

but n seems to identify separate specific solution. So we can write

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right]$$

Each amplitude A_p was written

$$A_p = C\sin p\theta \,, \quad \text{so write}$$

$$A_{pn} = C_n \sin\left(\frac{p\,n\pi}{N+1}\right)$$

where C_n is amplitude of mode n.

Displacement are then $\quad y_{pn}(t) = A_{pn}\cos(\omega_n t)$

How many modes should we expect for N masses?

Number of oscillators	Number of modes
1	1
2	2
⋮	
N	N?

Yes - N masses will have N modes.

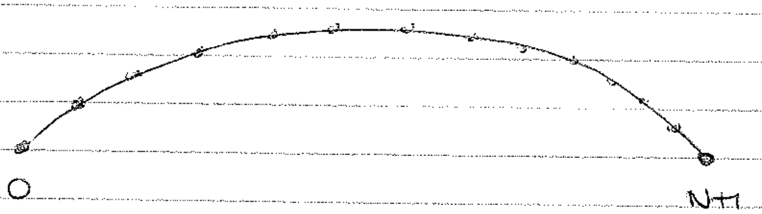
Basically, once you go beyond $n = N$ there are no new solutions - you start repeating ones you've already found. See p.142-143

Let's look at the modes:

1^{st} mode:

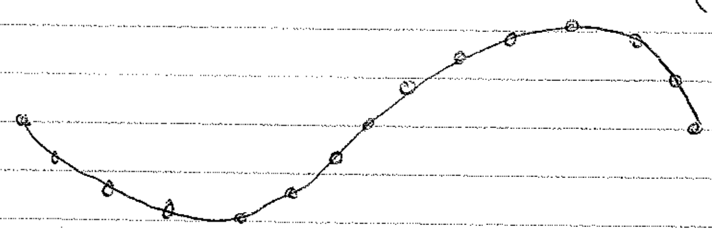
$$y_{p2} = \underbrace{C_1 \sin\left(\frac{p\pi}{N+1}\right)}_{\substack{\text{sets amplitudes} \\ \text{no time dependence}}} \underbrace{\cos \omega_1 t}_{\substack{\text{time dependence for} \\ \text{all oscillators}}}$$

as p goes from 1 to N then we have

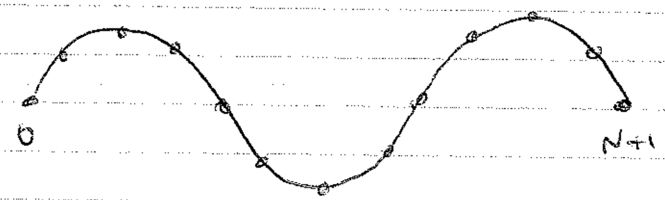


at $t=0$

For second mode $\quad y_{p2} = C_2 \sin\left(\frac{p2\pi}{N+1}\right) \cos \omega_2 t$



For 3^{rd} mode



etc.

Notes:

(1) Overall shape of mode is set by the sin term

(2) Frequency of the mode is set by n & $\omega_0 = \sqrt{\frac{T}{ml}}$

(3) No reason why solutions can not be added to get more complicated behaviors

(4) Amplitudes of different modes should be different and can be zero too.

(5) Note the curves we've drawn are just representative the string would be straight between masses.